CONNES' TRACE FORMULA AND DIRAC REALIZATION OF MAXWELL AND YANG-MILLS ACTION

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ABSTRACT. The paper covers known facts about the Dixmier trace (with some generalities about traces), the Wodzicki residue, and Connes' trace theorem, including two variants of proof of the latter. Action formulas are treated very sketchy, because they were considered in other lectures of the workshop.

CONTENTS

1. The Dixmier-Traces

The essential ingredients of the famous trace formula [3, 4] of A. Connes are operator algebraic constructs over the W^*-algebra $\mathsf{B}(\mathcal{H})$ of all bounded linear operators over some infinite dimensional separable Hilbert space $\mathcal{H}$ which are known as *Dixmier-traces*. The constructions will be explained in this section. For a general operator-algebraic background the reader is referred e.g. to [9, 31, 36, 22, 23].

1.1. **Generalities on traces on C^*- and W^*-algebras.**

1.1.1. *Basic topological notions, notations.* A C^*-algebra M is a Banach *-algebra, with *-operation $x \longmapsto x^*$ and norm $\| \cdot \|$ obeying $\|x^*x\| = \|x\|^2$, for each $x \in M$ (*-quadratic property). Let $M_+ = \{x^*x : x \in M\}$ be the cone of positive elements of M. For $x \in M$ let $x \geq 0$ be synonymous with $x \in M_+$. The Banach space of all continuous linear forms on M (dual) will be denoted by M^*, the dual norm (functional norm) be $\| \cdot \|_1$. As usual, a linear form f over M is termed positive, $f \geq 0$, if $f(x^*x) \geq 0$, for each $x \in M$. Remind that positive linear forms are automatically continuous and are generating for M^*. Thus $M_+^* = \{f \in M^* : f \geq 0\}$ is the set of all these forms. There is a fundamental result of operator theory saying that if M possesses a unit $\mathbf{1}$ (unital C^*-algebra), then $f \geq 0$ if, and only if, $f(\mathbf{1}) = \|f\|_1$. It is common use to refer to positive linear forms of norm one as states. Thus, in a unital C^*-algebra the set of all states on M, $\mathcal{S}(M)$, is easily seen to be a convex set which according to the Alaoglu-Bourbaki theorem is $\sigma(M^*, M)$-compact. Here, the $\sigma(M^*, M)$-topology (also w^*-topology in the special context at hand) is the weakest locally convex topology generated by the seminorms ρ_x, $x \in M$, with $\rho_x(f) = |f(x)|$, for each $f \in M^*$. The generalization of the notion of positive linear form on M (which refers to positive linear maps into the special C^*-algebra of complex numbers $\mathbb{C}$) is the notion of the positive linear map. Say that a linear mapping $T : M \longrightarrow N$ which acts from one C^*-algebra M into another one N is positive if $T(x^*x) \geq \mathbf{0}$ within N for each $x \in M$. In the unital case (for M) one then knows that $\|T(\mathbf{1})\| = \|T\|$ holds ($\|T\|$ refers to the operator norm of T as a linear operator acting from the one Banach space M into the other N). On the other hand, each linear map $T : M \longrightarrow N$ which obeys this relation is known to be positive. If both M and N have a unit, a linear map $T : M \longrightarrow N$ is said to be unital if $T(\mathbf{1}) = \mathbf{1}$ is fulfilled (the units being the respective units). Thus, and in particular, each unital linear map $T : M \longrightarrow N$ of norm one beween unital C^*-algebras has to be a positive map.

Remind that a C^*-algebra M is a W^*-algebra, that is, is *-isomorphic to some vN-algebra on some Hilbert space $\mathcal{H}$, if and only if, there exists a (unique) Banach subspace M_* (the predual space) of M^* such that M is the (continuous) dual of M_*, $M = (M_*)^*$. Note that a non-zero W^*-algebra is always unital. Suppose now that M is a W^*-algebra. The forms of M_* will be referred to as normal linear forms. One then knows that the normal positive linear forms $M_{*+} = M_* \cap M_+^*$ (resp. the normal states $\mathcal{S}_0(M) = M_* \cap \mathcal{S}(M)$) are generating for M_* in the sense that each normal linear form may be represented as a complex linear combination of at most four normal states. As a consequence of this, for each ascendingly directed (in the sense of $\leq$) bounded net $\{x_\alpha\} \subset M_+$ there exists a lowest upper bound $l.u.b. \, x_\alpha$ within M_+. On the other hand, a positive linear form $\omega \in M_+^*$ is normal if, and only if, for each ascendingly directed bounded net $\{x_\alpha\} \subset M_+$ the relation $\omega(l.u.b. \, x_\alpha) = l.u.b. \, \omega(x_\alpha) = \lim_\alpha \omega(x_\alpha)$ is valid. Note that in the latter

characterization it suffices if the mentioned continuity be fulfilled for ascendingly directed nets of orthoprojections of M.

In contrast to the previous, $\nu \in M_+^*$ is called singular if to each orthoprojection $p \in M$ with $\nu(p) > 0$ there is another orthoprojection $q \in M$ with $\mathbf{0} < q < p$ and $\nu(q) = 0$. According to [35] each $\omega \in M_+^* \backslash M_{+*}$ in a unique way can be decomposed as $\omega = \omega_1 + \omega_2$, with normal $\omega_1 \in M_{*+}$ and singular $\omega_2 \in M_+^*$.

The simplest example of a W^*-algebra where singular positive linear forms can exist is the commutative W^*-algebra $\ell^\infty = \ell^\infty(\mathbb{N})$ of all bounded sequences $x = (x_n) = (x_1, x_2, \dots)$ of complex numbers, with norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, and algebra multiplication $x \cdot y = (x_n y_n)$ and *-operation $x^* = (\bar{x}_n)$ defined componentwise. Recall that in this case the predual space ℓ_*^∞ is the Banach space of all absolutely summable sequences $\ell^1(\mathbb{N})$, with norm $\|\omega\|_1 = \sum_{n\in\mathbb{N}} |\omega_n|$ for $\omega = (\omega_n) \in \ell^1(\mathbb{N})(= \ell^1)$. Thereby, each such ω can be identified with an element in the dual Banach space $(\ell^\infty)^*$ via the identification with the linear functional $\omega(\cdot)$ given as $\omega(x) = \sum_{n\in\mathbb{N}} \omega_n x_n$, for each $x \in \ell^\infty$. For simplicity, also this functional $\omega(\cdot)$ will be referred to as ω, $\omega = \omega(\cdot)$.

In generalizing from the setting of a normal positive linear form, call a positive linear map $T : M \longrightarrow N$ from one W^*-algebra M into another W^*-algebra N normal if $T(l.u.b.\, x_\alpha) = l.u.b.\, T(x_\alpha)$ holds for each ascendingly directed bounded net $\{x_\alpha\} \subset M_+$. Note that in a W^*-algebra the Alaoglu-Bourbaki theorem may be applied on M, and then yields that the (closed) unit ball M_1 of M is $\sigma(M, M_*)$-compact. From this it follows that the unit ball within the bounded linear operators which map the W^*-algebra M into itself is compact with respect to the topology determined by the system of seminorms $\rho_{x,f}$, labelled by $x \in M$ and $f \in M_*$, and which are defined at T by $\rho_{x,f}(T) = |f \circ T(x)|$. Refer to this topology as the $\sigma(M, M_*)$-weak operator topology on the Banach algebra of bounded linear operators $\mathsf{B}(M)$ over the Banach space M. Thereby, by convention for $T, S \in \mathsf{B}(M)$ let the product $TS \in \mathsf{B}(M)$ be defined through successive application of maps to the elements of M in accordance with the rule 'apply right factor first', that is $TS(x) = (T \circ S)(x) = T(S(x))$.

1.1.2. *Traces on C^*- and W^*-algebras.* We recall the very basic facts on traces as found e.g. in [9, 6.1.]. A function $\tau : M_+ \longrightarrow \overline{\mathbb{R}}_+$ from the positive cone into the extended positive reals is said to be a *trace* provided it is (extended) *additive, positive homogeneous* (that is, $x, y \in M_+$ and $\lambda \in \mathbb{R}_+$ imply $\tau(x + y) = \tau(x) + \tau(y)$ and $\tau(\lambda x) = \lambda \tau(x)$, with $0 \cdot \infty = 0$ by convention) and obeys $\tau(x^*x) = \tau(xx^*)$, for each $x \in M$ (*invariance property*). ¿From the first two properties it follows that $M_{\tau+} = \{x \in M_+ : \tau(x) < \infty\}$ is a *hereditary subcone* of M_+, that is, $M_{\tau+}$ is a cone such that $x \in M_{\tau+}$ and $\mathbf{0} \le y \le x$ for $y \in M_+$ implies $y \in M_{\tau+}$. From this it is then easily inferred that $\mathcal{L}_\tau = \{x \in M : \tau(x^*x) < \infty\}$ is a *left ideal* in M. Owing to the invariance condition $\tau(x^*x) = \tau(xx^*)$ however, $\mathcal{L}_\tau$ has to be also a right ideal. Hence, $\mathcal{L}_\tau$ is a *two-sided ideal* of M, which is characteristic for the trace τ. It is known that the complex linear span $[M_{\tau+}]$ is a *-subalgebra of M which is even a two-sided ideal and which is generated by $\mathcal{L}_\tau$ as $[M_{\tau+}] = \mathcal{L}_\tau^2$. Also, on the *-subalgebra $[M_{\tau+}]$ there exists a unique (complex) linear form $\tilde{\tau}$ obeying $\tilde{\tau}(x) = \tau(x)$, for each $x \in M_{\tau+}$. Owing to invariance then also the characteristic commutation property $\tilde{\tau}(yx) = \tilde{\tau}(xy)$ holds, for each $x \in [M_{\tau+}]$ and all $y \in M$.

The two-sided ideal $[M_{\tau+}]$ will be referred to as *defining ideal* of the trace τ. Since the linear extension $\tilde{\tau}$ of τ from the cone of all positive elements of $[M_{\tau+}]$

(which according to the previous is $M_{\tau+}$) onto $[M_{\tau+}]$ is unique, by tacit understanding the notation $\tau(x)$ will be also used at non-positive x of the defining ideal of τ if the evaluation $\tilde{\tau}(x)$ of the linear functional $\tilde{\tau}$ at x is meant.

The trace τ is termed *finite trace* if $M_{\tau+} = M_+$, and *semifinite trace* if $\tau(x) = \sup\{\tau(y) : y \le x, \, y \in M_{\tau+}\}$, for each $x \in M_+$. If M is a W^*-algebra, with group of unitary elements $\mathcal{U}(M)$, the above condition on invariance usually is replaced with a seemingly weaker requirement upon *unitary invariance*, that is $\tau(u^*xu) = \tau(x)$ be fulfilled, for each $x \in M_+$ and $u \in \mathcal{U}(M)$. However, both conditions are equivalent there (and are so even on unital C^*-algebras). Also, in the W^*-case the trace τ is said to be *normal* provided for each ascendingly directed bounded net $\{x_\alpha\} \subset M_+$ the relation $\tau(l.u.b.\, x_\alpha) = l.u.b.\, \tau(x_\alpha) = \lim_\alpha \tau(x_\alpha)$ is fulfilled.
Now, suppose $I \subset M$ is a proper two-sided ideal of the W^*-algebra M. Then, I is also a *-subalgebra of M, with generating positive cone $I_+ = I \cap M_+$, that is, $I = [I_+]$ is fulfilled (these facts are consequences of the polar decomposition theorem, essentially). Under these premises we have the following extension principle:

Lemma 1.1. *Suppose I_+ is a hereditary subcone of M_+. Then, each additive, positive homogeneous and invariant map $\tau_0 : I_+ \longrightarrow \overline{\mathbb{R}}_+$ extends to a trace τ on M, with $M_{\tau+} = \{x \in I_+ : \tau_0(x) < \infty\}$.*

Proof. Define $\tau(x) = \tau_0(x)$, for $x \in I_+$, and $\tau(x) = \infty$ for $x \in M_+\backslash I_+$. Then, since I_+ is a hereditary cone, for $x, y \in M_+$ with $x + y \in I_+$ one has both $x, y \in I_+$, and thus $\tau(x) + \tau(y) = \tau(x+y)$ is evident from $\tau_0(x) + \tau_0(y) = \tau_0(x+y)$. For $x, y \in M_+$ with $x + y \notin I_+$ at least one of x, y must not be in I_+. Hence, $\tau(x) + \tau(y) = \infty$ and $\tau(x + y) = \infty$ by definition of τ. Thus additivity holds in any case. That τ is also positive homogeneous is clear. Finally, remind that, according to polar decomposition $x = u|x|$ for $x \in M$, one has $xx^* = u(x^*x)u^*$ and $x^*x = u^*(xx^*)u$, with the partial isometry $u \in M$. Since I is a two-sided ideal of M, from this one infers $xx^* \in I_+$ if, and only if, $x^*x \in I_+$. From this in view of the definition and since τ_0 is invariant on I_+ also invariance of τ on M_+ follows. $\qquad\square$

1.2. **Examples of traces.** In the following the classical special cases of traces on M with either $M = \mathsf{B}(\mathcal{H})$ or $M = \mathcal{C}\mathsf{B}(\mathcal{H})$ are considered in more detail, where $\mathcal{C}\mathsf{B}(\mathcal{H})$ is the C^*-subalgebra of $\mathsf{B}(\mathcal{H})$ of all *compact* linear operators on the separable infinite dimensional Hilbert space $\mathcal{H}$. For generalities on the theory of compact operators and proofs from there the reader is referred to [29, 32] e.g.; in the following recall only those few facts and details which are important in the context of traces. In all that follows, the scalar product $\mathcal{H} \times \mathcal{H} \ni \{\chi, \eta\} \longmapsto \langle \chi, \eta \rangle \in \mathbb{C}$ on $\mathcal{H}$ by convention is supposed to be linear with respect to the first argument χ, and antilinear in the second argument η, and maps into the complex field $\mathbb{C}$.

1.2.1. *Traces on compact linear operators.* We start with recalling the characterization of positive compact linear operators in terms of a spectral theorem. Let $x \in \mathsf{B}(\mathcal{H})_+$ be a non-trivial positive (=non-negative) bounded linear operator. Then, x is a (positive) compact operator, $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$ if, and only if, the following two condition are fulfilled. Firstly, there have to exist a non-increasing infinite sequence $(\mu_1(x), \mu_2(x), \ldots)$ of non-negative reals $\mu_k(x)$, which converge to zero as $k \to \infty$, and an infinite *orthonormal system* (o.n.s. for short) $\{\varphi_n\} \subset \mathcal{H}$ of eigenvectors of x obeying $x\varphi_k = \mu_k(x)\varphi_k$, for each $k \in \mathbb{N}$, and with $x\varphi = 0$, for each $\varphi \in [\{\varphi_n\}]^\perp$ (thus the spectrum of x is $\mathrm{spec}(x) = \{\mu_k(x) : k \in \mathbb{N}\} \cup \{0\}$).

And secondly, each non-zero eigenvalue of x has only finite multiplicity, that is, $m(\mu) = \#\{k : \mu_k(x) = \mu\}$ obeys $m(\mu) < \infty$, for each $\mu \in \mathbb{R}_+ \backslash \{0\}$.

Recall that $\mathcal{C}\mathsf{B}(\mathcal{H})$ is also a closed *-ideal of $\mathsf{B}(\mathcal{H})$. Hence, according to polar decomposition, $x \in \mathsf{B}(\mathcal{H})$ is compact if, and only if, the module $|x| = \sqrt{x^*x}$ of x is compact, $|x| \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$. In line with this and following some common use, for $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$ and in view of the above define $\mu_k(x) = \mu_k(|x|)$, for each $k \in \mathbb{N}$, and refer to the ordered sequence $\mu_1(x) \geq \mu_2(x) \geq \ldots$ of eigenvalues of $|x|$ (with each of the non-zero eigenvalues repeated according to its multiplicity) as *characteristic sequence* of x. The terms of this sequence can be obtained by minimizing the distance of the given compact operator x to the finite rank linear operators (which are special compact operators) of a fixed rank as follows:

$$\forall\, k \in \mathbb{N}: \ \mu_k(x) = \min\{\|x - y\| : y \in \mathcal{C}\mathsf{B}(\mathcal{H}),\, \dim y\mathcal{H} \leq k\}\,. \qquad (1.1a)$$

Alternatively, and yet more important, these values can be obtained also from a representation of the sequence $\{\sigma_n(x)\}$ of their partial sums $\sigma_k(x) = \sum_{j \leq k} \mu_k(x)$ which arises from maximizing the following expression over the unitaries $\mathsf{U}(\mathcal{H})$ of $\mathcal{H}$ and finite orthonormal systems $\{\psi_1, \ldots, \psi_k\} \subset \mathcal{H}$ of cardinality $k \in \mathbb{N}$:

$$\sigma_k(x) = \max\left\{\left|\sum_{j \leq k}\langle ux\psi_j, \psi_j\rangle\right| : u \in \mathsf{U}(\mathcal{H}),\, \{\psi_1, \ldots, \psi_k\} \text{ o.n.s.}\right\}, \qquad (1.1b)$$

which for positive x simplifies into

$$\forall\, x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+: \ \sigma_k(x) = \max\left\{\sum_{j \leq k}\langle x\psi_j, \psi_j\rangle : \{\psi_1, \ldots, \psi_k\} \text{ o.n.s.}\right\}. \qquad (1.1c)$$

Now, let us fix an arbitrary *maximal orthonormal system* (m.o.n.s. for short) $\{\varphi_n\} \subset \mathcal{H}$, and let p be an orthoprojection with $\dim p\mathcal{H} = k < \infty$. Then, for each $x \in \mathsf{B}(\mathcal{H})$ the operator xp is of finite rank, and for each o.n.s. $\{\psi_1, \ldots, \psi_k\}$ which linearily spans $p\mathcal{H}$, by elementary Hilbert space calculus one derives the relation

$$\sum_{n=1}^{\infty}\langle xp\varphi_n, \varphi_n\rangle = \sum_{j \leq k}\langle x\psi_j, \psi_j\rangle\,. \qquad (\times)$$

Hence, in case of compact x (1.1b) equivalently reads as

$$\sigma_k(x) = \max\left\{\left|\sum_{n=1}^{\infty}\langle uxp\varphi_n, \varphi_n\rangle\right| : u \in \mathsf{U}(\mathcal{H}),\, p = p^* = p^2,\, \dim p\mathcal{H} \leq k\right\}. \qquad (\star)$$

Note that for $x \geq \mathbf{0}$ the expression of $(\times)$ is positive and with the help of similarly elementary calculations as those which led to $(\times)$ one infers that for each orthoprojection p with $\dim p\mathcal{H} = k < \infty$ and any $x \in \mathsf{B}(\mathcal{H})_+$ the following holds:

$$\sum_{j \leq k}\langle x\psi_j, \psi_j\rangle = \sum_{n=1}^{\infty}\langle p\sqrt{x}\varphi_n, \sqrt{x}\varphi_n\rangle = \sum_{n=1}^{\infty}\langle xp\varphi_n, \varphi_n\rangle \geq 0\,. \qquad (\star\star)$$

Especially, since according to (1.1c) for *positive* compact x maximizing over the unitaries becomes redundant and may be omitted, in view of this and $(\star\star)$ for each such element the relation $(\star)$ then simplifies into the following well-known form:

$$\forall\, x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+: \ \sigma_k(x) = \max\left\{\sum_{n=1}^{\infty}\langle xp\varphi_n, \varphi_n\rangle : p = p^* = p^2,\, \dim p\mathcal{H} \leq k\right\}. \ (1.1d)$$

¿From (1.1b)–(1.1d) one now concludes some useful relations and estimates. The first is a rather trivial consequence of the definition of $\sigma_k(x)$ and says that

$$\forall\, x \in \mathcal{C}\mathsf{B}(\mathcal{H}),\ \lambda \in \mathbb{C},\ k \in \mathbb{N}:\ \sigma_k(\lambda x) = |\lambda|\, \sigma_k(x)\,. \tag{1.2a}$$

It is stated here only for completeness. In the special case of positive compact operators from (1.1c) we get the following often used estimates:

$$\forall\, x, y \in \mathcal{C}\mathsf{B}(\mathcal{H})_+,\ k \in \mathbb{N}:\ y \leq x \implies \sigma_k(y) \leq \sigma_k(x)\,. \tag{1.2b}$$

The next estimate is due to [25] and at once gets obvious from (1.1b), and tells us that the following holds:

$$\forall\, x, y \in \mathcal{C}\mathsf{B}(\mathcal{H}),\ k \in \mathbb{N}:\ \sigma_k(x + y) \leq \sigma_k(x) + \sigma_k(y)\,. \tag{1.2c}$$

The third estimate deals with an upper bound of $\sigma_k(x) + \sigma_k(y)$ in case of positive operators $x, y \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$ and arises from (1.1d). In line with the latter, let orthoprojections p, q of rank k be given such that $\sigma_k(x) = \sum_{n=1}^{\infty} \langle xp\varphi_n, \varphi_n \rangle$ and $\sigma_k(y) = \sum_{n=1}^{\infty} \langle yq\varphi_n, \varphi_n \rangle$ are fulfilled. Then, the least orthoprojection $p \vee q$ majorizing both p and q has rank $2k$ at most. Thus there is an orthoprojection Q of rank $2k$ and obeying $p \vee q \leq Q$. Hence, in view of the choice of p, q and with the help of $(\star\star)$ one infers that $\sum_{n=1}^{\infty} \langle xQ\varphi_n, \varphi_n \rangle = \sigma_k(x) + \sum_{n=1}^{\infty} \langle x(Q-p)\varphi_n, \varphi_n \rangle \geq \sigma_k(x)$ and $\sum_{n=1}^{\infty} \langle yQ\varphi_n, \varphi_n \rangle = \sigma_k(y) + \sum_{n=1}^{\infty} \langle x(Q-q)\varphi_n, \varphi_n \rangle \geq \sigma_k(y)$. In view of (1.1d) from this then $\sigma_k(x) + \sigma_k(y) \leq \sigma_{2k}(x + y)$ follows. For positive compact operators the previous together with (1.2c) may be summarized into the following one:

$$\forall\, x, y \in \mathcal{C}\mathsf{B}(\mathcal{H})_+,\ k \in \mathbb{N}:\ \sigma_k(x + y) \leq \sigma_k(x) + \sigma_k(y) \leq \sigma_{2k}(x + y)\,. \tag{1.2d}$$

Note that, since $\mathcal{C}\mathsf{B}(\mathcal{H})$ is a two-sided ideal, from (1.1a) for each $y \in \mathcal{C}\mathsf{B}(\mathcal{H})$ and $a, b \in \mathsf{B}(\mathcal{H})$ the estimate

$$\forall\, k \in \mathbb{N}:\ \mu_k(ayb) \leq \|a\|\, \|b\|\, \mu_k(y) \tag{1.2e}$$

can be obtained. Thus, under these conditions one has

$$\forall\, a, b \in \mathsf{B}(\mathcal{H}),\ y \in \mathcal{C}\mathsf{B}(\mathcal{H}),\ k \in \mathbb{N}:\ \sigma_k(ayb) \leq \|a\|\, \|b\|\, \sigma_k(y)\,. \tag{1.2f}$$

Especially, if $x = u|x|$ is the polar decomposition of $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$ within $\mathsf{B}(\mathcal{H})$, then with the partial isometry $u \in \mathsf{B}(\mathcal{H})$ one has both, $xx^* = ux^*xu^*$ and $x^*x = u^*xx^*u$. In the special cases of (1.2f) with $y = xx^*$, $a = u^*$, $b = u$ and $y = x^*x$, $a = u$, $b = u^*$ we arrive at estimates which fit together into the following assertion:

$$\forall\, x \in \mathcal{C}\mathsf{B}(\mathcal{H}),\ k \in \mathbb{N}:\ \sigma_k(x^*x) = \sigma_k(xx^*)\,. \tag{1.2g}$$

In the following, a trace τ is said to be *non-trivial* if there is at least one $x \geq \mathbf{0}$ with $0 < \tau(x) < \infty$. The relations given in eqs. (1.2) are the key facts that the theory of traces on both algebras $\mathcal{C}\mathsf{B}(\mathcal{H})$ and $\mathsf{B}(\mathcal{H})$ can be based on.

Lemma 1.2. *Let* $\mathrm{tr} : \mathcal{C}\mathsf{B}(\mathcal{H})_+ \longmapsto \overline{\mathbb{R}}_+$ *be defined by* $\mathrm{tr}\, x = \lim_{n \to \infty} \sigma_n(x)$, *for each* $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$. *Then,* tr *is a non-trivial semifinite trace on* $\mathcal{C}\mathsf{B}(\mathcal{H})$. *Moreover, to each non-trivial trace* τ *which does not vanish identically on the positive operators of finite rank there exists unique* $\lambda \in \mathbb{R}_+ \backslash \{0\}$ *such that* $\lambda \cdot \tau(x) \geq \mathrm{tr}\, x$ *holds for all* $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$, *and with equality occuring at each* x *of finite rank.*

Proof. The sequence $\{\sigma_n(x)\}$ is increasing, for each $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$. Thus $\mathrm{tr}\, x = \lim_{n \to \infty} \sigma_n(x)$ exists in the extended sense. Especially, from (1.2d) in the limit then additivity of tr follows, whereas from (1.2a) and (1.2g) homogeneity and invariance can be seen. Thus, tr is a trace (see 1.1.2). By construction $0 < \mathrm{tr}\, x < \infty$ for each compact positive $x \neq \mathbf{0}$ of finite rank. Thus tr is non-trivial. However, since $\mathcal{H}$

is infinite dimensional, $\operatorname{tr} x = \infty$ will occur for some positive compact operators. To see that tr is semifinite requires to prove that for $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$ with $\operatorname{tr} x = \infty$ there existed a sequence $\{x_n\} \subset \mathcal{C}\mathsf{B}(\mathcal{H})_+$ with $x_n \leq x$ and $\operatorname{tr} x_n < \infty$ such that $\lim_{n \to \infty} \operatorname{tr} x_n = \infty$. Note that by definition of tr, $\operatorname{tr} x = \infty$ implies that x cannot be of finite rank. Hence, x can be written as $x = \sum_{k=1}^{\infty} \mu_k(x) \, p_k$, with infinitely many mutually orthogonal one-dimensional orthoprojections p_k and all $\mu_k(x) \neq 0$. Clearly, for each $n \in \mathbb{N}$ the operators $x_n = \sum_{k=1}^{n} \mu_k(x) \, p_k$ are of finite rank and obey $\mathbf{0} \leq x_1 \leq x_2 \leq x_3 \leq \ldots \leq x$. Also, owing to $\sigma_k(x_n) = \sigma_n(x)$ for $k \geq n$, one has $\operatorname{tr} x_n = \sigma_n(x)$, and therefore $\lim_{n \to \infty} \operatorname{tr} x_n = \infty$ follows. Thus tr is semifinite. Suppose τ is a non-trivial trace. Thus $0 < \tau(y) < \infty$, for some positive compact y. Suppose $\tau(x) > 0$ for some $x \geq \mathbf{0}$ of finite rank. According to additivity and homogeneity of τ there has to exist a one-dimensional subprojection p of a spectral orthoprojection of x with $\tau(p) > 0$. The same arguments for y ensure that $\tau(q) < \infty$, for some one-dimensional subprojection q of some spectral projection of y. But since $q = vv^*$ and $p = v^*v$, with $v \in \mathcal{C}\mathsf{B}(\mathcal{H})$, by invariance of τ one has $\tau(q) = \tau(p)$. Hence $\infty > \tau(p) > 0$, and $\tau(q) = \tau(p)$ for each one-dimensional orthoprojection q. Put $\lambda = \tau(p)^{-1}$. Then $\lambda \cdot \tau(q) = \operatorname{tr} q$, and thus $\lambda \cdot \tau(x) = \operatorname{tr} x$ for each positive operator x of finite rank. Finally, if $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$ is not of finite rank, let $\mathbf{0} \leq x_1 \leq x_2 \leq x_3 \leq \ldots \leq x$ be the above approximating sequence of x by finite rank operators x_n. Also in such case $\lim_{n \to \infty} \operatorname{tr} x_n = \lim_{n \to \infty} \sigma_n(x) = \operatorname{tr} x$ follows. Hence, in view of the above relation over the operators of finite rank, and since $\tau(x_n) \leq \tau(x)$ holds, $\operatorname{tr} x = \lim_{n \to \infty} \operatorname{tr} x_n = \lambda \lim_{n \to \infty} \tau(x_n) \leq \lambda \cdot \tau(x)$. $\square$

For completeness, we give yet the most famous formula relating tr and which makes that this trace is so extremely useful.

Corollary 1.1. *For each maximal orthonormal system $\{\psi_n\} \subset \mathcal{H}$ and $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$ one has $\operatorname{tr} x = \sum_{n=1}^{\infty} \langle x\psi_n, \psi_n \rangle$.*

Proof. Let $\{\varphi_j\}$ be an o.n.s. with $x\varphi_k = \mu_k(x)\varphi_k$, for all $k \in \mathbb{N}$, and be p_n the orthoprojection with $p_n\mathcal{H} = [\varphi_1, \ldots, \varphi_n]$. Then, by positivity of x one has $\langle x\psi_n, \psi_n \rangle = \langle \sqrt{x}\psi_n, \sqrt{x}\psi_n \rangle \geq \langle p_k\sqrt{x}\psi_n, \sqrt{x}\psi_n \rangle$, and therefore and in view of $(\star\star)$ one gets $\sum_{n=1}^{\infty} \langle x\psi_n, \psi_n \rangle \geq \sum_{n=1}^{\infty} \langle p_k\sqrt{x}\psi_n, \sqrt{x}\psi_n \rangle = \sum_{j=1}^{k} \langle x\varphi_j, \varphi_j \rangle = \sigma_k(x)$. According to Lemma 1.2 then $\sum_{n=1}^{\infty} \langle x\psi_n, \psi_n \rangle \geq \operatorname{tr} x$ follows. On the other hand, if q_k is the orthoprojection onto $[\psi_1, \ldots, \psi_k]$, according to (1.1d) for each $k \in \mathbb{N}$ certainly $\sum_{n=1}^{k} \langle x\psi_n, \psi_n \rangle = \sum_{n=1}^{\infty} \langle xq_k\psi_n, \psi_n \rangle \leq \sigma_k(x)$. From this in view of Lemma 1.2 once more again $\sum_{n=1}^{\infty} \langle x\psi_n, \psi_n \rangle \leq \operatorname{tr} x$ is seen. Taking together this with the above estimate provides that equality has to occur. $\square$

A non-zero trace τ on $\mathcal{C}\mathsf{B}(\mathcal{H})$ will be said to be *singular* if $\tau(x) = 0$ for each $x \geq \mathbf{0}$ of finite rank. Relating this and non-trivial traces there is the following result.

Corollary 1.2. *Let τ be a non-trivial trace on $\mathcal{C}\mathsf{B}(\mathcal{H})$. Then, either $\tau = \lambda \cdot \operatorname{tr}$ holds, for a unique $\lambda \in \mathbb{R}_+$, or there exist a singular trace τ_s and a unique $\alpha \in \mathbb{R}_+$ such that $\tau = \tau_s + \alpha \cdot \operatorname{tr}$.*

Proof. If $\tau = \lambda \cdot \operatorname{tr}$ is fulfilled, then $\tau(p) = \lambda \operatorname{tr} p$, for each one-dimensional orthoprojection p. Owing to $\operatorname{tr} p = 1$ (see Corollary 1.1) then $\lambda = \tau(p)$ follows.

Suppose $\tau \notin \mathbb{R}_+ \operatorname{tr}$. Then, $\tau \neq 0$, and if a decomposition $\tau = \tau_s + \alpha \cdot \operatorname{tr}$ with singular τ_s exists, then $\tau(p) = \alpha$, for some (and thus any) one-dimensional orthoprojection p, and the following two alternatives have to be dealt with: firstly, if τ is vanishing on all positive operators of finite rank, τ is singular, and $\tau = \tau_s$

and $\alpha = 0$ have to be chosen (see above). Secondly, if τ does not vanish on all positive operators of finite rank, according to Lemma 1.2 there exists unique $\lambda > 0$ with $\lambda \cdot \tau(x) \geq \operatorname{tr} x$, for each $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$, with equality occuring on any operator of finite rank. Hence, in defining $\tau_s(x) = \tau(x) - \lambda^{-1} \operatorname{tr} x$, for each x with $\operatorname{tr} x < \infty$, and $\tau_s(x) = \infty$ else, we get a positive map τ_s which does not vanish identically on the positive compact operators, but which is vanishing on all positive operators of finite rank. From the previous and since both τ and tr are traces, also additivity, positive homogeneity and invariance of τ_s at once follow. Hence, τ_s is a singular trace, which is easily seen to obey $\tau = \tau_s + \alpha \cdot \operatorname{tr}$, with $\alpha = \lambda^{-1}$. $\square$

1.2.2. *Traces on* $\mathsf{B}(\mathcal{H})$. Remind in short the theory of traces on $M = \mathsf{B}(\mathcal{H})$, with separable infinite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{F}\mathsf{B}(\mathcal{H})$ be the two-sided ideal of all operators of finit rank in $\mathsf{B}(\mathcal{H})$. In the following an ideal $\mathcal{I}$ will be termed *non-trivial* if $\mathcal{I} \neq \{\mathbf{0}\}$ and $\mathcal{I} \neq \mathsf{B}(\mathcal{H})$. Both $\mathcal{F}\mathsf{B}(\mathcal{H})$ and $\mathcal{C}\mathsf{B}(\mathcal{H})$ are non-trivial two-sided ideals. Thereby, the compact operators form a closed ideal, with $\mathcal{F}\mathsf{B}(\mathcal{H})$ being dense within $\mathcal{C}\mathsf{B}(\mathcal{H})$. Start with a useful criterion on non-compactness for a positive operator.

Lemma 1.3. *A positive operator $x \geq \mathbf{0}$ is non-compact if, and only if, there exist real $\lambda > 0$ and infinite dimensional orthoprojection p obeying $\lambda p \leq x$.*

Proof. Note that, in contrast to the spectral characterization of positive compact operators, the spectral theorem in case of a *non-compact* $x \geq \mathbf{0}$ with $\# \operatorname{spec}(x) < \infty$ provides that $\lambda p \leq x$ has to be fulfilled, for some non-zero λ and orthoprojection p with $\dim p\mathcal{H} = \infty$ (for one $\lambda \in \operatorname{spec}(x) \backslash \{0\}$ at least the corresponding spectral eigenprojection p has to meet the requirement). But then, due to normclosedness of the compact operators, and since for each positive x one has $x \in \overline{\{y : \mathbf{0} \leq y \leq x, \# \operatorname{spec}(x) < \infty\}}$ (uniform closure), such type of estimate has to exist in each case of a non-compact positive operator x. On the other hand, if $\lambda p \leq x$ is fulfilled, for some non-zero λ and infinite dimensional orthoprojection p, in view of this relation the equivalence of p with the unit operator $\mathbf{1}$ will imply $v^* x v$ to be invertible, for the partial isometry v achieving $p = vv^*$, $\mathbf{1} = v^*v$. Thus, owing to the non-triviality of the ideal $\mathcal{C}\mathsf{B}(\mathcal{H})$, $v^* x v \notin \mathcal{C}\mathsf{B}(\mathcal{H})$ has to hold. Due to two-sidedness of $\mathcal{C}\mathsf{B}(\mathcal{H})$ the latter requires that also x was non-compact. $\square$

Corollary 1.3. *Both $\mathcal{F}\mathsf{B}(\mathcal{H})_+$ and $\mathcal{C}\mathsf{B}(\mathcal{H})_+$ are hereditary subcones of $\mathsf{B}(\mathcal{H})_+$.*

Proof. For $\mathcal{F}\mathsf{B}(\mathcal{H})_+$ the assertion is trivial. For non-zero $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$ and positive $y \neq \mathbf{0}$ with $y \leq x$ also y must be compact since otherwise the criterion of Lemma 1.3 were applicable to y with resulting in a contradiction to the assumed compactness of x, by the same criterion. $\square$

The following is likely the most remarkable result relating ideals in $\mathsf{B}(\mathcal{H})$ and descends from [1], see also [32, Lemma 11, Theorem 11].

Theorem 1.1. $\mathcal{F}\mathsf{B}(\mathcal{H}) \subset \mathcal{I} \subset \mathcal{C}\mathsf{B}(\mathcal{H})$, *for each non-trivial, two-sided ideal $\mathcal{I}$.*

As a consequence of this the defining ideal of a non-trivial trace τ on $\mathsf{B}(\mathcal{H})$ always is a non-zero ideal of compact operators. Thus especially $\tau(x) = \infty$ must be fulfilled, for each $x \in \mathsf{B}(\mathcal{H})_+ \backslash \mathcal{C}\mathsf{B}(\mathcal{H})_+$. On the other hand, since according to Corollary 1.3 $\mathcal{C}\mathsf{B}(\mathcal{H})_+$ is a hereditary cone, whenever τ_0 is a non-zero trace on $\mathcal{C}\mathsf{B}(\mathcal{H})$, then the extension principle of Lemma 1.1 can be applied and shows that

upon defining $\tau(x) = \tau_0(x)$ for $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$, and $\tau(x) = \infty$ for $x \in \mathsf{B}(\mathcal{H})_+ \backslash \mathcal{C}\mathsf{B}(\mathcal{H})_+$, a non-zero trace τ on $\mathsf{B}(\mathcal{H})$ is given. Thus, traces (resp. non-trivial traces) on all bounded linear operators are in one-to-one correspondence with traces (resp. non-trivial traces) on the compact operators.

For the unique extension of the trace tr of Lemma 1.2 from compact operators onto $\mathsf{B}(\mathcal{H})$ the same notation tr will be used. Note that in view of Lemma 1.3 with the help of 1.2.1 ($\times$) and ($\star\star$) easily follows that for non-compact $x \geq \mathbf{0}$ and each m.o.n.s. $\{\varphi_n\}$ one has $\sum_{n=1}^{\infty} \langle x\varphi_n, \varphi_n \rangle = \infty$. Hence, the formula given in Corollary 1.1 extends on all $x \in \mathsf{B}(\mathcal{H})_+$. From this formula it is plain to see that tr is a non-trivial *normal* trace on $\mathsf{B}(\mathcal{H})$. Up to a positive multiple, tr is also unique on $\mathsf{B}(\mathcal{H})$ as non-trivial trace with this property :

Corollary 1.4. *A non-trivial normal trace τ has the form $\tau = \alpha \cdot \mathrm{tr}$, with $\alpha > 0$.*

Proof. Let $p_1 < p_2 < p_3 < \ldots < \mathbf{1}$ be a sequence of orthoprojections with $\mathrm{rank}(p_n) = n$, for each $n \in \mathbb{N}$. Then, for each $x \geq \mathbf{0}$, $l.u.b.\sqrt{x}\,p_n\sqrt{x} = x$. Note that $x_n = \sqrt{x}\,p_n\sqrt{x} \in \mathcal{F}\mathsf{B}(\mathcal{H})_+$ holds. Since also $\tau|_{\mathcal{C}\mathsf{B}(\mathcal{H})_+}$ is a non-trivial trace, by Corollary 1.2 there is unique $\alpha > 0$ with $\tau(x_n) = \alpha \cdot \mathrm{tr}\, x_n$, for each $n \in \mathbb{N}$. Hence, by normality of τ and since tr is normal, $\tau(x) = \alpha \cdot \mathrm{tr}\, x$ follows, for each $x \geq \mathbf{0}$. $\quad\square$

Note that in view of the mentioned one-to-one correspondence with traces on the compact operators Corollary 1.2 extends to non-trivial traces on $\mathsf{B}(\mathcal{H})$ accordingly. In line with this and Corollary 1.4 the theory of traces on $\mathsf{B}(\mathcal{H})$ with separable infinite dimensional $\mathcal{H}$ essentially is the theory of the one normal trace tr and myriads of singular traces.

1.3. **Examples of singular traces on $\mathsf{B}(\mathcal{H})$.** Examples of singular traces have been invented by J. Dixmier in [8]. Nowadays this class is referred to as Dixmier-traces. In the following, only the singular traces of this class will be constructed and considered. Thereby, in constructing these traces we will proceed in two steps.

In a first step we are going to define some non-trivial two-sided ideal in $\mathsf{B}(\mathcal{H})$, with hereditary positive cone, which later will prove to belong to the defining ideal of each of the singular traces to be constructed. As has been already noticed in context of Theorem 1.1, each such ideal then is an ideal of compact operators. For such ideals one knows that these can be completely described in terms of the classes (Schatten-classes) of the characteristic sequences coming along with the operators of the ideal, see [32, Theorem 12]. In these sequences, which are in $\ell^\infty(\mathbb{N})_+$, the full information on the ideal is encoded.

In a second step, a class of states on $\ell^\infty(\mathbb{N})$ is constructed which, in restriction to the mentioned sequences from the ideal, yields a map which vanishes on those sequences which correspond to operators of finite rank. If taken as functions on the positive operators of the ideal these maps will be shown to be additive, positive homogeneous and invariant. Hence, the extension via the extension principle of Lemma 1.1 on all of $\mathsf{B}(\mathcal{H})_+$ finally will provide us with a class of singular traces.

1.3.1. *Step one*: *Some ideal of compact operators.* For compact x with the help of the characteristic sequence $\{\mu_n(x)\}$ define

$$\forall\, k \in \mathbb{N}\backslash\{1\}: \ \gamma_k(x) = \frac{1}{\log k} \sum_{j \leq k} \mu_j(x) = \frac{\sigma_k(x)}{\log k}\,. \tag{1.3a}$$

Then, $\{\gamma_n(x) :\ n > 1\}$ is a sequence of non-negative reals which may be bounded or not. The bounded situation deserves our special interest. Let a subset $L^{1,\infty}(\mathcal{H}) \subset \mathcal{C}\mathsf{B}(\mathcal{H})$ be defined as follows:

$$L^{1,\infty}(\mathcal{H}) = \left\{ x \in \mathcal{C}\mathsf{B}(\mathcal{H}) :\ \sup_{n \geq 2} \gamma_n(x) < \infty \right\}. \qquad (1.3\mathrm{b})$$

It is plain to see that by $L^{1,\infty}(\mathcal{H})$ an ideal is given in $\mathsf{B}(\mathcal{H})$, for some corresponding terminology see [28, 3, 4], and e.g. [14].

Proposition 1.1. $L^{1,\infty}(\mathcal{H})$ *is a non-trivial two-sided ideal in* $\mathsf{B}(\mathcal{H})$, *and thus is an ideal of compact operators, with hereditary cone* $L^{1,\infty}(\mathcal{H})_+$ *of positive elements.*

Proof. In view of the definitions (1.3) and since $\mathcal{C}\mathsf{B}(\mathcal{H})$ is a two-sided ideal, the validity of the first assertion follows as an immediate consequence of (1.2a), (1.2c) and (1.2f) together with the fact that for each operator x of finite rank $\{\gamma_n(x) :\ n > 1\}$ is a null-sequence and thus is bounded. Finally, owing to Corollary 1.3 for $x \in L^{1,\infty}(\mathcal{H})_+$ and $y \in \mathsf{B}(\mathcal{H})$ with $\mathbf{0} \leq y \leq x$ one infers $y \in \mathcal{C}\mathsf{B}(\mathcal{H})_+$, and then $y \leq x$ according to (1.2b) implies also $y \in L^{1,\infty}(\mathcal{H})_+$. $\qquad\square$

For completeness yet another characterization of $L^{1,\infty}(\mathcal{H})$ will be noted (without proof, see e.g. in [4, IV.2.β]), and a class of $L^{1,\infty}$-elements, which can be characterized through the asymptotic behavior of the singular values, will be given.

Let $L^1(\mathcal{H})$ be the ideal of all operators of *trace-class*, that is, the defining ideal which corresponds to the normal trace tr, cf. Lemma 1.2 and Corollary 1.1. From (1.3b) and Lemma 1.2 then especially follows that the inclusion relation $L^1(\mathcal{H}) \subset L^{1,\infty}(\mathcal{H})$ takes place amongst $L^{1,\infty}(\mathcal{H})$ and the ideal of trace-class operators. Moreover, if in line with [28] another Banach space $L^{\infty,1}(\mathcal{H})$ ($= \mathfrak{S}_\omega$ in [28]) is defined through

$$L^{\infty,1}(\mathcal{H}) = \left\{ y \in \mathcal{C}\mathsf{B}(\mathcal{H}) :\ \sum_{n=1}^{\infty} n^{-1} \mu_n(y) < \infty \right\}, \qquad (1.4)$$

then it is essentially due to (1.2c), (1.2e) and by monotonicity of the sequences of the $\sigma_n(y)$'s and $\frac{1}{n}$'s that also $L^{\infty,1}(\mathcal{H})$ is a non-trivial two-sided ideal (Macaev-ideal). Note that in analogy to the above also in this case obviously an inclusion with trace-class operators takes place, $L^1(\mathcal{H}) \subset L^{\infty,1}(\mathcal{H})$. The ideals from (1.3b) and (1.4) are related by the duality given through the 2-form $\Omega(x,y) = \mathrm{tr}\, xy$. Namely, each $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$ obeying $xy \in L^1(\mathcal{H})$, for all $y \in L^{\infty,1}(\mathcal{H})$, is in $L^{1,\infty}(\mathcal{H})$.

Proposition 1.2. $L^{1,\infty}(\mathcal{H})$ *is the dual to the Macaev-ideal.*

Also, in this context note that for each $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$ obeying $xz \in L^1(\mathcal{H})$ for all $z \in I$, with an ideal I of compact operators, and each $y \in I$ the relation

$$|\Omega(x,y)| \leq \sum_{n=1}^{\infty} \mu_n(x)\,\mu_n(y) < \infty \qquad (1.5\mathrm{a})$$

must be fulfilled. In fact, since by assumption for $a, b \in \mathsf{B}(\mathcal{H})$ also $axby \in L^1(\mathcal{H})$ is fulfilled, in view of the polar decomposition of x, y the estimate $|\Omega(x,y)| \leq \sup_{u,v} |\,\mathrm{tr}\, u|x|v|y|\,|$ can be easily inferred, with u, v extending over the partial isometries in $\mathsf{B}(\mathcal{H})$. Also, with the help of Corollary 1.1, and (1.1b) e.g., one finds that $\sup_{u,v} |\,\mathrm{tr}\, u|x|v|y|\,| \leq \sup_{\vec{r}} \sum_{n=1}^{\infty} \mu_n(x) r_n$ must hold, with $\vec{r} = (r_1, r_2, \dots)$ obeying $r_1 \geq r_2 \geq r_3 \geq \dots \geq 0$ and $\sum_{k \leq n} r_k \leq \sigma_n(y) = \sum_{k \leq n} \mu_k(y)$, for each $n \in \mathbb{N}$

(the ordering of $\mu_n(x)$'s is of importance in this context). Since also the sequence of $\mu_n(y)$'s is in decreasing order, it is not hard to see that by successively exploiting the just mentioned conditions on $\vec{r}$, for $n \leq N$ with $N \in \mathbb{N}$, the validity of $\sum_{k \leq N} \mu_k(x)(\mu_k(y) - r_k) \geq 0$ can be derived, for each $N \in \mathbb{N}$, and any given $\vec{r}$ which is subject to the above conditions. From this the left-hand side estimate of (1.5a) gets evident. Now, for any two given compact linear operators x, y in view of the polar decomposition theorem and owing to compactness of both operators partial isometries u, w can be chosen such that $u|x|w|y| \geq \mathbf{0}$ holds, with the singular values of the compact operator $u|x|w|y|$ obeying $\mu_n(u|x|w|y|) = \mu_n(x)\,\mu_n(y)$, for each $n \in \mathbb{N}$. In accordance with Lemma 1.2 one then has $|\operatorname{tr} u|x|v|y|\,| = \operatorname{tr} u|x|w|y| = \lim_{n\to\infty} \sigma_n(u|x|w|y|) = \sum \mu_n(u|x|w|y|) = \sum \mu_n(x)\,\mu_n(y)$. Hence, since in our particular situation of x, y we have $u|x|w|y| \in L^1(\mathcal{H})$ and the above proved left-hand side estimate of (1.5a) has been shown to hold, (1.5a) is completely seen.

Especially, in view of Proposition 1.2 the estimate (1.5a) can be applied with $x \in L^{1,\infty}(\mathcal{H})$ and $I = L^{\infty,1}(\mathcal{H})$. Relating asymptotic properties of singular values of $x \in L^{1,\infty}(\mathcal{H})$ we thus get the following information:

$$x \in L^{1,\infty}(\mathcal{H}) \implies \forall\, y \in L^{\infty,1}(\mathcal{H}) : \sum_{n=1}^{\infty} \mu_n(x)\,\mu_n(y) < \infty. \qquad (1.5b)$$

Viewing (1.4) and (1.5b) together suggests compact x with asymptotic behavior of singular values like $\mu_n(x) = \mathbf{O}(n^{-1})$ as good candidates for elements of $L^{1,\infty}(\mathcal{H})$.[1] In fact, such asymptotic behavior implies that, with some $C > 0$, for all $n \geq 2$

$$\sigma_n(x) = \sum_{1 \leq k \leq n} \mu_k(x) \leq C\left\{1 + \sum_{2 \leq k \leq n} k^{-1}\right\} \leq C\left\{1 + \int_1^n t^{-1} dt\right\} = C(1 + \log n)$$

is fulfilled. In view of (1.3) we therefore arrive at the following result:

Corollary 1.5. $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$, $\mu_n(x) = \mathbf{O}(n^{-1}) \implies x \in L^{1,\infty}(\mathcal{H})$.

Remark 1.1.   (1) It is easy to see that for compact x with bounded multiplicity function, $m(\lambda) \leq N < \infty$ for all λ, the condition imposed by (1.5b) upon x amounts to $\mu_n(x) = \mathbf{O}(n^{-1})$. Unfortunately, in case of unbounded m this can fail to hold.[2] That this can even occur for x within $L^{1,\infty}(\mathcal{H})$ can be seen by the following counterexample:[3]

(2) Let x be positive and compact with $\mu_1(x) = 1$, and with singular values which for $k \geq 2$ with $(m-1)! < k \leq m!$, $m \geq 2$, are given by $\mu_k(x) = \log m / m!$. One then easily proves that $\sigma_k(x)$ obeys $\sigma_{m!}(x) \leq 1 + \log m!$. Since the function $f(t) = \log(1 + t/m!) - t\{\log(m+1)/(m+1)!\}$ is non-negative for $0 \leq t \leq m \cdot m!$, from the previous also $\sigma_k(x) \leq 1 + \log k$ can be followed whenever $m! < k \leq (m+1)!$ is fulfilled. This conclusion applies for each $m \geq 2$, and thus according to (1.3) we finally get $x \in L^{1,\infty}(\mathcal{H})$. On the other hand, $\lim_{m\to\infty} m!\,\mu_{m!}(x) = \infty$ holds. Thus in particular $\mu_n(x)$ certainly cannot behave asymptotically like $\mathbf{O}(n^{-1})$.

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[1] As usual, for $g : \mathbb{N} \ni n \mapsto g(n) \in \mathbb{R}_+ \backslash \{0\}$ the notation $x_n = \mathbf{O}(g(n))$ is a shorthand notation for $|x_n| < C\,g(n)$, with some $C > 0$ (and accordingly defined with $\mathbb{R}_+$ instead of $\mathbb{N}$).

[2] We are grateful to C. Portenier for mentioning this fact to us.

[3] The counterexample has been communicated to us by J. Várilly, see also [15, Lemma 7.35].

1.3.2. *Step two*: *Scaling invariant states.* Let us come back now to the construction of the Dixmier-traces. The construction will be based on considering a certain class of states on the commutative W*-algebra $M = \ell^\infty$, see 1.1.1 for basic notations. Relating special further notations, for each $k \in \mathbb{N}$ let $e_k \in \ell^\infty$ be the k-th atom in ℓ^∞, with j-th component obeying $(e_k)_j = \delta_{kj}$ (Kronecker symbol), and let E_k be the special orthoprojection of rank k given as $E_k = \sum_{j \le k} e_j$. The ascendingly directed sequence $\{E_n\}$ obeys $l.u.b.\, E_n = \mathbf{1}$ and the following equivalence is valid:

$$x \in \ell^\infty : \ \| \cdot \|_\infty - \lim_{n \to \infty} E_n x = x \iff \lim_n x_n = 0. \qquad (1.6)$$

Also, for $x \in \ell^\infty_+$, $\{E_n x\} \subset \ell^\infty_+$ is ascendingly directed, with $l.u.b.\, E_n x = x$.

For the following, let a mapping $\mathbf{s} : \ell^\infty \longrightarrow \ell^\infty$ (scaling) be defined on $x \in \ell^\infty$ through $\mathbf{s}(x)_j = x_{2j}$, for all $j \in \mathbb{N}$. It is obvious that $\mathbf{s}$ is a normal *-homomorphism onto ℓ^∞. Hence, $\mathbf{s}$ is a unital normal positive linear map onto itself, and $\ell^\infty_{\mathbf{s}} = \{x \in \ell^\infty : \mathbf{s}(x) = x\}$ is a W*-subalgebra of ℓ^∞ (the fixpoint algebra of $\mathbf{s}$).

Lemma 1.4. *There exists a conditional expectation* $\mathcal{E} : \ell^\infty \longrightarrow \ell^\infty_{\mathbf{s}}$ *projecting onto the fixpoint algebra* $\ell^\infty_{\mathbf{s}}$ *such that the following properties hold*:

(1) $\mathcal{E} \circ \mathbf{s} = \mathcal{E}$;
(2) $\mathcal{E}(x) = (\lim_{n \to \infty} x_n) \cdot \mathbf{1}$, *for each* $x \in \ell^\infty$ *with* $\lim_{n \to \infty} x_n$ *existing.*

Proof. Let us consider the sequence $\{\mathbf{s}\langle n \rangle\}$ of partial averages $\mathbf{s}\langle n \rangle = \frac{1}{n} \sum_{k \le n} \mathbf{s}^k$, $n \in \mathbb{N}$. Since these all are unital positive linear maps, by $\sigma(\ell^\infty, \ell^1)$-weak compactness of the closed unit ball in $\mathsf{B}(\ell^\infty)$ the sequence of partial averages then must have a $\sigma(\ell^\infty, \ell^1)$-weak cluster point $\mathcal{E}$ which has to be a unital positive linear map, too. Since then $\mathcal{E} = \sigma(\ell^\infty, \ell^1) - \text{weak} \lim_\lambda \mathbf{s}\langle n_\lambda \rangle$ has to be fulfilled for some appropriately chosen subnet $\{\mathbf{s}\langle n_\lambda \rangle\}$, the inclusion $\ell^\infty_{\mathbf{s}} \subset \{x \in \ell^\infty : \mathcal{E}(x) = x\}$ gets evident. Since $\mathbf{s}\langle n \rangle \circ \mathbf{s} = \mathbf{s} \circ \mathbf{s}\langle n \rangle$ and $\|\mathbf{s}\langle n \rangle \circ \mathbf{s} - \mathbf{s}\langle n \rangle\| \le \frac{2}{n}$ hold, for each $n \in \mathbb{N}$, and since owing to normality of $\mathbf{s}$ for each $\omega \in \ell^1$ also $\omega \circ \mathbf{s} \in \ell^1$ is fulfilled, by argueing with the mentioned subnet one infers that $\mathcal{E} \circ \mathbf{s} = \mathbf{s} \circ \mathcal{E} = \mathcal{E}$. From this $\{x \in \ell^\infty : \mathcal{E}(x) = x\} \subset \ell^\infty_{\mathbf{s}}$ and $\mathbf{s}\langle n \rangle \circ \mathcal{E} = \mathcal{E}$ follow, for each n. Thus in view of the above $\mathcal{E}^2 = \mathcal{E} \circ \mathcal{E} = \mathcal{E}$ follows. Hence, $\mathcal{E}$ is a projection of norm one (conditional expectation) projecting onto the fixpoint algebra of $\mathbf{s}$ and which satisfies (1).

To see (2), note first that owing to $\mathbf{s}(e_k) = \mathbf{0}$ for k odd, and $\mathbf{s}(e_k) = e_{k/2}$ for k even, one certainly has $\mathbf{s}^n(E_k) = \mathbf{0}$, for each $n > \log k / \log 2$. Hence, the action of the n-th average $\mathbf{s}\langle n \rangle$ to the orthoprojection E_k can be estimated as $\|\mathbf{s}\langle n \rangle(E_k)\|_\infty \le [\log k / \log 2]/n$ (here $[\cdot]$ means the integer part), and thus for all $k \in \mathbb{N}$ one has $\| \cdot \|_\infty - \lim_{n \to \infty} \mathbf{s}\langle n \rangle(E_k) = \mathbf{0}$. ¿From this and $\mathcal{E} = \sigma(\ell^\infty, \ell^1) - \text{weak} \lim_\lambda \mathbf{s}\langle n_\lambda \rangle$ then especially $\omega(\mathcal{E}(E_k)) = 0$ follows, for each $\omega \in \ell^1$. Hence $\mathcal{E}(E_k) = \mathbf{0}$, for each k. Since for each $y \in \ell^\infty$ with $\mathbf{0} \le y \le \mathbf{1}$ one has $\mathbf{0} \le E_k y \le E_k$, from the previous together with positivity of $\mathcal{E}$ also $\mathcal{E}(E_k y) = \mathbf{0}$ follows. By linearity of $\mathcal{E}$ and since ℓ^∞ is the linear span of $\ell^\infty_+ \cap (\ell^\infty)_1$ this remains true for each $y \in \ell^\infty$. But then, for $x \in \ell^\infty$ with $\alpha = \lim_{n \to \infty} x_n$ by continuity of $\mathcal{E}$ and in view of (1.6) one infers $\mathcal{E}(x - \alpha \cdot \mathbf{1}) = \| \cdot \|_\infty - \lim_{k \to \infty} \mathcal{E}(E_k(x - \alpha \cdot \mathbf{1})) = \mathbf{0}$, which is equivalent with (2). $\square$

Corollary 1.6. *There is a state* $\omega \in \mathcal{S}(\ell^\infty)$ *satisfying the following properties*:

(1) $\omega \circ \mathbf{s} = \omega$;
(2) $\omega(x) = \lim_{n \to \infty} x_n$, *provided* $\lim_{n \to \infty} x_n$ *exists.*

The set $\Gamma_{\mathsf{s}}(\ell^{\infty})$ *of all such states is a* w^*-*compact convex subset of singular states.* [4]

Proof. Let $\mathcal{E}$ be constructed as in Lemma 1.4. By positivity and unitality of $\mathcal{E}$, for each $\nu \in \mathcal{S}(\ell^{\infty})$ also $\omega = \nu \circ \mathcal{E}$ is a state. In view of (1)–(2) this state then obviously satisfies (1)–(2).[5] That $\Gamma_{\mathsf{s}}(\ell^{\infty})$ is w^*-compact and convex is evident from the linear nature of the conditions (1)–(2). Finally, in accordance with (2) one has $\omega(E_k) = 0$, for each $\omega \in \Gamma_{\mathsf{s}}(\ell^{\infty})$ and all $k \in \mathbb{N}$. Now, let $p \in \ell^{\infty}$ be any orthoprojection with $\omega(p) > 0$. Then, $p \neq \mathbf{0}$, and owing to $l.u.b. \, E_n p = p$ there has to exist $k \in \mathbb{N}$ with $q = E_k p \neq \mathbf{0}$. Thus $\mathbf{0} < q < p$ and $q \leq E_k$. In view of the above from the latter by positivity of ω then $\omega(q) = 0$ follows. Hence, each $\omega \in \Gamma_{\mathsf{s}}(\ell^{\infty})$ is singular. $\square$

1.3.3. *Constructing the Dixmier-traces.* For given $x \in L^{1,\infty}(\mathcal{H})$, let a sequence $\gamma(x)$ be given through $\gamma(x) = (\gamma_2(x), \gamma_3(x), \dots)$, with $\gamma_n(x)$ in accordance with (1.3a). Then, by definition (1.3b) one has $\gamma(x) \in \ell^{\infty}_+$. Hence, if for each fixed scaling invariant state $\omega \in \Gamma_{\mathsf{s}}(\ell^{\infty})$, see Corollary 1.6, following [8] we define

$$\forall \, x \in L^{1,\infty}(\mathcal{H})_+ \; : \; \mathrm{Tr}_{\omega}(x) = \omega(\gamma(x)) \,, \tag{1.7}$$

then according to Proposition 1.1 and since ω is a positive linear form, we are given a *positive* map $\mathrm{Tr}_{\omega} : L^{1,\infty}(\mathcal{H})_+ \ni x \longmapsto \mathrm{Tr}_{\omega}(x) \in \mathbb{R}_+$ defined on the positive cone of the ideal $L^{1,\infty}(\mathcal{H})$. The key idea of [8] is that additivity of Tr_{ω} can be shown.

Lemma 1.5. Tr_{ω} *is an additive, positive homogeneous and invariant map from* $L^{1,\infty}(\mathcal{H})_+$ *into* $\mathbb{R}_+$.

Proof. Since $L^{1,\infty}(\mathcal{H})_+$ is the positive cone of a two-sided ideal of compact operators, for $x, y \in L^{1,\infty}(\mathcal{H})_+$ and $\lambda \in \mathbb{R}_+$ we have that $x + y, \, \lambda x, \, x^* x, \, xx^* \in L^{1,\infty}(\mathcal{H})_+$, and these are compact operators again. Hence, in view of (1.3a) from (1.2a) and (1.2g) both $\lambda \cdot \gamma(x) = \gamma(\lambda x)$ and $\gamma(x^* x) = \gamma(xx^*)$ follow, which in line with (1.7) means that Tr_{ω} is positive homogeneous and invariant. It remains to be shown that Tr_{ω} is additive. First note that according to the left-hand side estimate of (1.2d) within ℓ^{∞}_+ one has $\gamma(x + y) \leq \gamma(x) + \gamma(y)$. Hence, by positivity and linearity of ω, (1.7) yields

$$\mathrm{Tr}_{\omega}(x + y) \leq \mathrm{Tr}_{\omega}(x) + \mathrm{Tr}_{\omega}(y) \,. \tag{$\star$}$$

Now, to each compact operator z let $\gamma^0(z) = (\gamma_3(z), \gamma_4(z), \dots)$, that is, $\gamma^0(z)$ arises from $\gamma(x)$ by application of the one-step left-shift. Also, on ℓ^{∞} let a linear map $\mathbf{m}$ be defined by $\mathbf{m}(\beta)_n = \frac{\log 2}{\log(n+1)} \cdot \beta_n$, for all $n \in \mathbb{N}$, at $\beta \in \ell^{\infty}$. One then has

$$\forall \, \beta \in \ell^{\infty} \; : \; \lim_{n \to \infty} \mathbf{m}(\beta)_n = 0 \,. \tag{$\star\star$}$$

Note that $\gamma^0(z) \in \ell^{\infty}$ whenever $z \in L^{1,\infty}(\mathcal{H})$. We are going to estimate $\gamma(z) - \gamma^0(z)$ for $z \in L^{1,\infty}(\mathcal{H})$. Since both $\{\sigma_n(z)\}$ and $\{\log n\}$ are monotoneously increasing, in view of the definition (1.3a) for each $z \in L^{1,\infty}(\mathcal{H})$ the follwing estimates at once can be seen to hold, for all $k \in \mathbb{N} \setminus \{1\}$:

$$\gamma_k(z) - \gamma_{k+1}(z) \leq \gamma_{k+1}(z) \left(\frac{\log(k+1)}{\log k} - 1 \right) \leq \frac{\log 2}{\log k} \cdot \gamma_{k+1}(z) \leq \frac{\log 2}{\log k} \, \|\gamma(z)\|_{\infty} \,.$$

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[4]Let another map $\mathbf{d}$ (doubling) over ℓ^{∞} be defined at x by $\mathbf{d}(x)_j = x_{[(1+j)/2]}$, $j \in \mathbb{N}$ ($[r]$ refers to the integer part of r). Then, scaling is left-inverse to doubling in $\mathsf{B}(\ell^{\infty})$, and thus in addition to (1) one also has $\mathbf{d}$-invariance of each $\omega \in \Gamma_{\mathsf{s}}(\ell^{\infty})$ as well.

[5]The usage of Lemma 1.4 might be avoided in this context; as we learned from [33] a positivity and separation argument of Hahn-Banach type may be used instead as well.

On the other hand, we also have

$$\gamma_k(z) - \gamma_{k+1}(z) \geq \frac{\sigma_k(z) - \sigma_{k+1}(z)}{\log(k+1)} = -\frac{\mu_{k+1}(z)}{\log(k+1)} \geq -\frac{1}{\log(k+1)} \cdot \|z\|.$$

¿From these two estimates we infer that $\Delta(z) = \gamma(z) - \gamma^0(z)$ for $z \in L^{1,\infty}(\mathcal{H})$ is a null-sequence in ℓ^∞, that is, $\lim_{n\to\infty} \Delta(z)_n = 0$ is fulfilled. According to the choice of ω and in accordance with Corollary 1.6 (2) we thus have the following to hold :

$$\forall z \in L^{1,\infty}(\mathcal{H}),\, \omega \in \Gamma_{\mathsf{s}}(\ell^\infty):\ \omega(\gamma(z)) = \omega(\gamma^0(z))\,. \qquad (\star\star\star)$$

Let us come back to our above $x, y \in L^{1,\infty}(\mathcal{H})_+$. Having in mind the definitions of the positive linear operators s and m as well as the meanings of γ and γ^0, it is easily inferred that from the right-hand side estimate in (1.2d) when divided by $\log k$, and considered for all $k \geq 2$, the estimate $\gamma(x) + \gamma(y) \leq (\sigma_{2k}(x+y)/\log k) = \mathsf{s}(\gamma^0(x+y)) + \mathsf{m} \circ \mathsf{s}(\gamma^0(x+y))$ can be followed to hold in ℓ_+^∞. By positivity and linearity of ω from this then

$$\omega(\gamma(x)) + \omega(\gamma(y)) \leq \omega \circ \mathsf{s}(\gamma^0(x+y)) + \omega \circ \mathsf{m}(\mathsf{s}(\gamma^0(x+y))) \qquad (\circ)$$

follows. Now, in view of $(\star\star)$ and Corollary 1.6 (2) one has $\omega \circ \mathsf{m}(\mathsf{s}(\gamma^0(x+y))) = 0$, whereas from Corollary 1.6 (1) and $(\star\star\star)$ one concludes that $\omega \circ \mathsf{s}(\gamma^0(x+y)) = \omega(\gamma(x+y))$. These facts together with $(\circ)$ fit together into the estimate $\omega(\gamma(x)) + \omega(\gamma(y)) \leq \omega(\gamma(x+y))$, which in view of (1.7) says that $\mathrm{Tr}_\omega(x) + \mathrm{Tr}_\omega(y) \leq \mathrm{Tr}_\omega(x+y)$ has to be valid. The latter and $(\star)$ then make that the desired additivity $\mathrm{Tr}_\omega(x) + \mathrm{Tr}_\omega(y) = \mathrm{Tr}_\omega(x+y)$ holds. $\qquad\square$

1.3.4. *The Dixmier-trace as a singular trace.* We remind that according to Proposition 1.1 the positive cone of the ideal $L^{1,\infty}(\mathcal{H})$ is hereditary. Thus the extension principle of Lemma 1.1 according to Lemma 1.5 for each $\omega \in \Gamma_{\mathsf{s}}(\ell^\infty)$ allows to extend the map Tr_ω of (1.7) to a trace on $\mathsf{B}(\mathcal{H})$. Thereby, the extension constructed in accordance with the proof of Lemma 1.1 will be the unique one with defining ideal $L^{1,\infty}(\mathcal{H})$. For this trace the same notation Tr_ω will be used henceforth. We refer to this trace as *Dixmier-trace* (to the particular $\omega \in \Gamma_{\mathsf{s}}(\ell^\infty)$). The essential properties of Dixmier-traces are summarized in the following.

Theorem 1.2. Tr_ω *is a singular trace on* $\mathsf{B}(\mathcal{H})$, *for each* $\omega \in \Gamma_{\mathsf{s}}(\ell^\infty)$. *The following properties are fulfilled*:

(1) $L^{1,\infty}(\mathcal{H})_+ = \{x \in \mathsf{B}(\mathcal{H})_+ :\ \mathrm{Tr}_\omega(x) < \infty\}\,;$
(2) $x \in L^{1,\infty}(\mathcal{H})_+,\, \exists \lim_{n\to\infty} \gamma_n(x) \implies \mathrm{Tr}_\omega(x) = \lim_{n\to\infty} \gamma_n(x)\,.$

Proof. The validity of (1) follows since the traces in question all are obtained as extensions of the maps given in (1.7), which satisfy Lemma 1.5 and which have range $\mathbb{R}_+$ (and not merely $\overline{\mathbb{R}}_+$). Since each state $\omega \in \Gamma_{\mathsf{s}}(\ell^\infty)$ obeys Corollary 1.6 (2), in view of the previous and (1.7) also (2) follows. Finally, for each $x \in \mathcal{F}\mathsf{B}(\mathcal{H})_+$ the sequence $\gamma(x)$ is a null-sequence, and therefore especially $x \in L^{1,\infty}(\mathcal{H})_+$, and as a special case of (2) then $\mathrm{Tr}_\omega(x) = 0$ follows. Hence, Tr_ω is a singular trace. $\qquad\square$

Note the remarkable feature of the Dixmier-traces coming along with Theorem 1.2 (2) and saying that provided certain circumstances are fulfilled for x, e.g. if the sequence $\{\gamma_n(x)\}$ has a limit, then independent of the state-parameter ω all these Dixmier-traces may yield the same common value at this x. It is such case of independence one usually is tacitly addressing to when speaking simply of *the* Dixmier-trace of x, whereas the operator itself then is referred to as *measurable*

operator, cf. [4, IV.2, Definition 7]. Some criteria of measurability, which however all reduce upon showing that the above mentioned special case of existence of $\lim_{n\to\infty} \gamma_n(x)$ would happen, subsequently will be discussed in more detail.

1.4. **Calculating the Dixmier-trace.**

1.4.1. *Simple criteria of measurability.* We start with discussing conditions which read in terms of spectral theory and which ensure that – for a given operator $x \in L^{1,\infty}(\mathcal{H})$ which is not simply of finite rank – the above-mentioned special case of measurability occurs, that is, the limit $\lim_{n\to\infty} \gamma_n(x)$ exists. As a first result of that kind one has the following one:[6]

Lemma 1.6. *Suppose $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$, with $\mu_n(x) \sim L \cdot n^{-1}$. Then $\lim_{n\to\infty} \gamma_n(x) = L$.[7]*

Proof. For compact operator x suppose $\lim_{n\to\infty} n \mu_n(x) = L$ to be fulfilled. Then, in case of $L > 0$, for δ with $L > \delta > 0$, let $M(\delta) \in \mathbb{N}$ be chosen such that

$$\forall n \geq M = M(\delta): \ (L-\delta)\, n^{-1} \leq \mu_n(x) \leq (L+\delta)\, n^{-1} \,.$$

¿From this for each $n \geq M$ we get

$$(L-\delta) \sum_{M < k \leq n} k^{-1} \leq \sum_{M < k \leq n} \mu_n(x) \leq (L+\delta) \sum_{M < k \leq n} k^{-1} \,. \qquad (\times)$$

Since $0 < t \mapsto t^{-1}$ is a strictly monotone decreasing function and the sequence of the singular values is decreasingly ordered, with the help of

$$\int_{M+1}^{n} dt\, t^{-1} \leq \sum_{M < k \leq n} k^{-1} \leq \int_{M}^{n} dt\, t^{-1}$$

which holds for $n > M$ the above estimate $(\times)$ implies

$$\int_{M+1}^{n} dt\, (L-\delta)\, t^{-1} \leq \sigma_n(x) - \sigma_M(x) \leq \int_{M}^{n} dt\, (L+\delta)\, t^{-1} \,.$$

¿From this for all $n > M = M(\delta)$

$$(L-\delta)\{1 - \log(M+1)/\log n\} \leq \gamma_n(x) - \sigma_M(x)/\log n \leq (L+\delta)\{1 - \log M/\log n\}$$

is obtained. Considering these estimates for $n \to \infty$ then yields

$$(L-\delta) \leq \liminf_{n\to\infty} \gamma_n(x) \leq \limsup_{n\to\infty} \gamma_n(x) \leq L + \delta \,.$$

Note that in case of $L = 0$ by positivity of all $\gamma_n(x)$ instead of the previous one finds $0 \leq \liminf_{n\to\infty} \gamma_n(x) \leq \limsup_{n\to\infty} \gamma_n(x) \leq \delta$, for any $\delta > 0$. Thus, since $\delta > 0$ can be chosen arbitrarily small, in either case $\lim_{n\to\infty} \gamma_n(x) = L$ follows. $\square$

Now, let us suppose $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$, with $\mu_n(x) = \mathbf{O}(\frac{1}{n})$. According to Corollary 1.5 we even have $x \in L^{1,\infty}(\mathcal{H})$, and since $\sum_n n^{-(1+\varepsilon)} < \infty$ is fulfilled for each $\varepsilon > 0$, then $|x|^z$ at each $z \in \mathbb{C}$ with $\Re z > 1$ has to be of trace-class and the definition

$$\zeta_x(z) = \sum_{n \geq 1} \mu_n(x)^z = \operatorname{tr} |x|^z \qquad (1.8)$$

will provide us with some holomorphic function ζ_x in the half-plane $\Re z > 1$. For this modification of the Riemann ζ-function the following holds.

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[6] We are grateful to C. Portenier, Marburg, for suggesting some details around this and related subjects [30].

[7] For $f : \mathbb{N} \to \mathbb{R}_+$ and $g : \mathbb{N} \to \mathbb{R}_+ \backslash \{0\}$ the notation $f(n) \sim L \cdot g(n)$ stands for $\lim_{n\to\infty} f(n)/g(n) = L$ (and accordingly defined with $\mathbb{R}_+$ instead of $\mathbb{N}$).

Lemma 1.7. *Let $x \in \mathcal{C}\mathsf{B}(\mathcal{H})$, with $\mu_n(x) = \mathbf{O}(\frac{1}{n})$. Suppose ζ_x admits an extension onto the half-plane $\Re z \geq 1$ which is continuous there except for a simple pole with residue L at $z = 1$, at worst. Then even $\mu_n(x) \sim L \cdot n^{-1}$ holds.*

Proof. In case of $x \in \mathcal{F}\mathsf{B}(\mathcal{H})$ one has $\lim_n n \cdot \mu_n(x) = 0$ as well as $\lim_{\varepsilon \to 0+} \operatorname{tr} |x|^{1+\varepsilon} = \operatorname{tr}|x| = \sum_n \mu_n(x) < \infty$, by triviality. From the latter $\lim_{s \to 1+}(s-1)\zeta_x(s) = 0$ follows. Hence, for each operator x of finite rank the assertion is true, with $L = 0$.

Suppose now that x is not of finite rank, $x \notin \mathcal{F}\mathsf{B}(\mathcal{H})$. In view of definitions (1.3a) and (1.8), upon possibly considering instead of x a scaling λx by a suitably chosen real $\lambda > 0$, without loss of generality it suffices if the assertion for $x \geq \mathbf{0}$ with $\mu_1(x) < 1$ can be shown. In line with this assume such $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+ \backslash \mathcal{F}\mathsf{B}(\mathcal{H})$.

By the spectral theorem there exists a spectral representation of x as an operator Stieltjes-integral $x = \int_0^{1^-} \lambda \, E(d\lambda)$, with projection-valued measure $E(d\lambda)$ derived from a left-continuous spectral family $\{E(\lambda) : \lambda \in \mathbb{R}\}$, that is, a family of orthoprojections obeying $E(t) \leq E(\lambda)$, for $t \leq \lambda$, $E(s) = \mathbf{0}$, for $s \leq 0$ and $E(\lambda-) = l.u.b._{t<\lambda}E(t) = E(\lambda)$, for each $\lambda \leq \infty$, with $l.u.b._{t<\infty}E(t) = \mathbf{1}$. By convention, for $a < b$, then $\int_a^{b-} E(d\lambda) = E(b) - E(a) = E([a,b[)$ and $\int_a^b E(d\lambda) = E(b+) - E(a) = E([a,b])$, and so on accordingly, where e.g. $E(b+)$ stands for the greatest lower bound $E(b+) = g.l.b._{t>b}E(t)$.

By means of some functional calculus and owing to normality of the trace tr it is easily inferred that (1.8) can be represented as an ordinary Stieltjes integral:

$$\forall z \in \mathbb{C}, \Re z > 1 \; : \; \zeta_x(z) = \int_{1+}^{\infty} t^{-z} \, d\alpha(t), \qquad (1.9a)$$

with the monotone increasing function α given by

$$\alpha(t) = \operatorname{tr} E([1/t, \infty[). \qquad (1.9b)$$

But then, if the assumptions on ζ_x are fulfilled with $\lim_{s \to 1+}(s-1)\zeta_x(s) = L$, all conditions for an application of Ikehara's theorem [21], are given (we refer to the formulation in [40, THEOREM 16]). In line with this the conclusion is that asymptotically

$$\alpha(t) \sim L \cdot t \qquad (1.9c)$$

has to be fulfilled as t tends to infinity. Since x is a compact operator, in view of the properties of the spectral resolution E together with normality of tr the definition (1.9b) provides a right-continuous, integral-valued step function which is constant between inverses of neighbouring spectral values of x. Especially, in case of $n \in \mathbb{N}$ with $\mu_n(x) > \mu_{n+1}(x)$ one infers that α for all t with $\mu_n(x)^{-1} \leq t < \mu_{n+1}(x)^{-1}$ yields $\alpha(t) = n$. A moments reflection then shows that with respect to each term of the ordered sequence $n_1 < n_2 < n_3 < \ldots$ of all subscripts where the value of μ_n jumps the relation (1.9c) in view of $\lim_{n \to \infty} \mu_n(x) = 0$ and by continuity of the parameter t in particular also implies both $\lim_{k \to \infty} n_{k+1} \, \mu_{n_{k+1}}(x) = L$ and $\lim_{k \to \infty} n_k \, \mu_{n_{k+1}}(x) = L$ to be fulfilled. But then, since $\mu_n(x) = \mu_{n_{k+1}}(x)$ holds for $n_k < n \leq n_{k+1}$, also $\lim_{n \to \infty} n \, \mu_n(x) = L$ can be obtained from these limit relations. Thus under the condition of the hypothesis also $\mu_n(x) \sim L \cdot n^{-1}$ in case of $x \geq \mathbf{0}$ and which is not of finite rank. In accordance with our preliminary remarks the assertion then has to be true, in either case under the mentioned hypothesis. $\square$

Remark 1.2. (1) Relating Lemma 1.6 remark that there are examples of operators where $\lim_{n\to\infty} \gamma_n(x) = L$ exists but $\mu_n(x) \not\sim L \cdot n^{-1}$, see [42, Beispiel A.27] or [15, Lemma 7.35].

(2) On the one hand, the conditions imposed on $\mu_n(x)$ and ζ_x in Lemma 1.7 simply reproduce the usual conditions for the standard results of Tauberian type[8] to become applicable. On the other hand, that the behavior of the extension of ζ_x at the whole line $\Re z = 1$ (and not only at $z = 1$) has to be of relevance can be seen also by example: there is $x \in \mathcal{CB}(\mathcal{H})$ with $\mu_n(x) = \mathbf{O}(\frac{1}{n})$ and $\lim_{s\to 1+}(s-1)\zeta_x(s) = 1$ but for which $\mu_n(x) \not\sim 1/n$.[9]

1.4.2. *A residue-formula for the Dixmier-trace.* The most important from practical point of view special case of measurability for an operator x occurs if the limit $\lim_n \gamma_n(x)$ exists. In particular, according to the previous considerations the latter will happen e.g. provided some function-theoretic assumptions on x can be satisfied. In these cases a formula arises which allows us to calculate the (singular) Dixmier-trace with the help of the ordinary trace as a limit of some function-theoretic expression of the operator in question. In fact, in view of Theorem 1.2 (2) and upon combining Lemma 1.7 and Lemma 1.6 we get the following result:

Corollary 1.7. *For each $x \in \mathcal{CB}(\mathcal{H})_+$ with $\mu_n(x) = \mathbf{O}(\frac{1}{n})$ one has $x \in L^{1,\infty}(\mathcal{H})_+$, and then by $\zeta_x(z) = \operatorname{tr} x^z$ a holomorphic function in the half-plane $\Re z > 1$ is given. Suppose ζ_x extends onto the half-plane $\Re z \geq 1$ and is continuous there except for a simple pole at $z = 1$, at worst. Then the Dixmier-trace of x is obtained as*

$$\operatorname{Tr}_\omega(x) = \lim_n \gamma_n(x) = \lim_{s\to 1+}(s-1)\operatorname{tr} x^s. \tag{1.10a}$$

Especially, when ζ_x extends to a meromorphic function on the whole complex plane, with a simple pole at $z = 1$ at worst, this formula turns into

$$\operatorname{Tr}_\omega(x) = \mathfrak{Res}|_{z=1}(\zeta_x), \tag{1.10b}$$

with the residue $\mathfrak{Res}(\zeta_x)$ of the extended complex function, taken at $z = 1$.

For completeness remark that by our Corollary 1.7, which is sufficient to cope with our later needs around Connes' trace theorem, in the special cases at hand the implication (1) $\Rightarrow$ (2) of [4, IV, Proposition 4] is reproduced.

Clearly, from both the theoretical and practical point of view, in context of the previous those situations deserve the main interest where formula (1.10b) could be applied. According to the results in [17, THEOREM 7.1, 7.2] this happens e.g. if the context of the classical pseudodifferential operators of order $-n$ acting on the sections $\Gamma(E)$ of a complex vector bundle $E \to M$ of a n-dimensional compact Riemannian manifold M is considered.

In fact, in [17] one proves that as a consequence of the good function-theoretic properties of ζ_x for each such operator the Weyl's formula of the asymptotic distribution of the spectral values [43] can be seen to hold. Thus, in particular the condition $\mu_k(x) = \mathbf{O}(1/k)$ is then fulfilled automatically and does not appear as an independent condition any longer.

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[8]This especially concerns theorems of Hardy and Littlewood [19] and Ikehara [21], see [18, Chap. VII, 7.5] and [40, see especially on p. 126 and THEOREM 18].

[9]J. Várilly has informed us about this fact and examples and counterexamples around this question which will appear in [15]. Also we are very indebted to J. Várilly for some clarifying remarks and hints to the literature.

But then, upon combining formula (1.10b) with a method [17, THEOREM 7.4, 7.5] (or see [44]) of expressing the residue in terms of the principal symbol of the classical pseudodifferential operator in question, one finally will arrive at Connes' trace theorem.

2. THE CONNES' TRACE THEOREM AND ITS APPLICATION

In the following we are going to comment on the way along to Connes' trace theorem in a more detailed manner and will give some indications on applications of this formula as to classical Yang-Mills theory.

2.1. Preliminaries.

2.1.1. *Basic facts about pseudodifferential operators.* Let Ω be an open set in $\mathbb{R}^n$, and let $C_0^\infty(\Omega)$ be the space of smooth functions with compact support inside Ω.

Definition 2.1. Let $p \in C^\infty(\Omega \times \mathbb{R})$. p is called a symbol of order (at most) $m \in \mathbb{R}$, if it satisfies the estimates

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{\alpha\beta K}(1 + \|\xi\|)^{m-|\alpha|}, \ x \in K, \ \xi \in \mathbb{R}^n, \tag{2.1}$$

for any choice of multiindices α, β and compact $K \subset \Omega$. The space of the symbols of order m is denoted by $S^m(\Omega \times \mathbb{R}^n)$ or simply S^m.

Note that our definition corresponds to the special case with $\varrho = 1$ and $\delta = 0$ of a more general class of symbols as considered e.g. in [34, Definition 1.1.], to which and to [10, 11] the reader might refer also for other details on pseudodifferential operators.[10] It is obvious that $S^m \subset S^k$ for $m \le k$. For $p \in S^m$, let $p(x, D)$ denote the operator

$$(p(x,D)u)(x) = (2\pi)^{-n/2} \int p(x,\xi) e^{i\langle x,\xi\rangle} \hat{u}(\xi) d\xi. \tag{2.2}$$

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-i\langle x,\xi\rangle} u(x) dx \tag{2.3}$$

is the Fourier transform of u. Note that different $p, p' \in S^m$ may lead to the same operator, $p(x, D) = p'(x, D)$.

Definition 2.2. A pseudodifferential operator (ψDO) of order (at most) m is an operator of the form

$$P = p(x, D), \tag{2.4}$$

where $p \in S^m$. The class of ψDO's of order m is denoted by L^m.

The mapping $S^m \longrightarrow L^m$, $p \mapsto p(x, D)$, is surjective, but in general it will not be injective. Its kernel is contained in $S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m$. The ψDO's corresponding to $S^{-\infty}$ form the space $L^{-\infty}$ of smoothing operators[11]. The principal symbol $\sigma_m(P)$ of a ψDO P of order m with symbol $p \in S^m$ is the class of p in S^m/S^{m-1}.

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[10]Relating notions, conventions and terminology, we do not follow the usage of [34] into any detail, but instead join some slightly simplified conventions and notations which are suitable for our purposes and which we borrowed from some survey lectures of E. Zeidler [45], and which are the same as in [12] and [16, 10.4., especially § 10.4.7.].

[11]For a more precise argumentation the reader is referred to [34, §3.2, Definition 3.3 and *Remark*], and where also the notion of 'properly supported ψDO' comes into play, but which for simplicity will not be considered explicitely in this paper.

Definition 2.3. $p \in S^m$ is called classical, if it has an "asymptotic expansion"

$$p \sim \sum_{j=0}^{\infty} p_{m-j}, \tag{2.5}$$

i.e. $p_{m-j} \in S^{m-j}$ and

$$p - \sum_{j=0}^{N-1} p_{m-j} \in S^{m-N}, \ \forall N, \tag{2.6}$$

and if p_{m-j} is positive homogeneous in ξ "away from 0", i.e.

$$p_{m-j}(x, t\xi) = t^{m-j} p_{m-j}(x, \xi), \ \|\xi\| \geq 1, \ t \geq 1. \tag{2.7}$$

A ψDO is said to be classical if its symbol is classical. The spaces of classical symbols and ψDOs are denoted by S_{cl}^m and L_{cl}^m respectively.

Let $p_{m-j}^0(x, \xi)$ be homogeneous functions in ξ on $\Omega \times (\mathbb{R}^n \setminus \{0\})$ coinciding with p_{m-j} for $\|\xi\| \geq 1$. These functions are uniquely determined, and one writes also

$$p \sim \sum_{j=0}^{\infty} p_{m-j}^0 \tag{2.8}$$

instead of (2.5). The principal symbol of a classical ψDO can be identified with the leading term p_m^0 in the asymptotic expansion (2.8).

Theorem 2.1. *Let $F : \Omega' \longrightarrow \Omega$ be a diffeomorphism of domains in $\mathbb{R}^n$. Then to every ψDO P on Ω with symbol $p \in S^m(\Omega \times \mathbb{R}^n)$ corresponds a ψDO P' on Ω' with symbol $p' \in S^m(\Omega' \times \mathbb{R}^n)$ such that:*

$$F^*(Pu) = P'(F^*(u)), \ u \in C_0^{\infty}(\Omega), \ F^* - pull\text{-}back, \tag{2.9}$$

$$p'(x, \xi) - p(F(x), ({}^tF'(x))^{-1}\xi) \in S^{m-1}(\Omega' \times \mathbb{R}^n). \tag{2.10}$$

If P is a classical ψDO then so is P'.

The theorem makes it possible to define ψDO's on manifolds. Let M be a paracompact smooth manifold, and consider an operator $A : C_0^{\infty}(M) \longrightarrow C^{\infty}(M)$. If Ω is some coordinate neighborhood of M, there are a natural extension map $i_{\Omega} : C_0^{\infty}(\Omega) \longrightarrow C_0^{\infty}(M)$ and a natural restriction map $p_{\Omega} : C^{\infty}(M) \longrightarrow C^{\infty}(\Omega)$. A is called ψDO of order m if all the local restrictions $A_{\Omega} := p_{\Omega} \circ A \circ i_{\Omega} : C_0^{\infty}(\Omega) \longrightarrow C^{\infty}(\Omega)$ are ψDO of order m. By Theorem 2.1, this is a good definition, and also classical ψDO can be defined in this manner. Moreover, equation (2.10) says that the principal symbol has an invariant meaning as a function on the cotangent bundle T^*M.
On the other hand, ψDO on a manifold can be constructed by gluing: Let $\bigcup_j \Omega_j = M$ be a locally finite covering of M by coordinate neighbourhoods, and let A_j be ψDO's of order m on Ω_j. Furthermore, let $\sum_j \psi_j = 1$ be a partition of unity subordinate to the given covering, and let $\phi_j \in C_0^{\infty}(\Omega_j)$ with $\phi_j|_{supp \ \psi_j} = 1$. Then $A := \sum_j \phi_j \circ A_j \circ \psi_j$ (ϕ_j, ψ_j considered as multiplication operators) is a ψDO of order m on M whose restrictions A_{Ω_j} coincide with A_j.

ψDO's acting on sections of vector bundles are defined with appropriate modifications: They are glued from local ψDO's which are defined using matrices of

symbols. The principal symbol is then a function on T^*M with values in the endo-morphisms of E, i.e. a section of the bundle $\pi^*(End(E))$, where $\pi : T^*M \longrightarrow M$ is the projection of the cotangent bundle, and $End(E)$ is the bundle of endomorphisms of E.

ψDO's are operators from $C_0^\infty(M)$ to $C^\infty(M)$. ψDO's of order m can be extended to bounded linear operators $H^s(M) \longrightarrow H^{s-m}(M)$, $s \in \mathbb{R}$ (Sobolev spaces). Notice that, by the Sobolev embedding theorems, every ψDO of order ≤ 0, $H^s \longrightarrow H^{s-m}$, can be considered as an operator $H^s \longrightarrow H^s$. In particular, taking the case $s = 0$, every ψDO of order ≤ 0 may be considered as an operator $L^2 \longrightarrow L^2$. For the case of manifolds, a Riemannian metric is used in the definition of the L^2 scalar products, for vector bundles in addition a fibre metric. $L^2(M, E)$ denotes the corresponding space of L^2 sections. We will need the following list of facts (for some terminology and the corresponding generalities see [34, Definition 3.1., 24.3] and [7, 23.26.12.] e.g.):
1. The product (which exists, if at least one of the factors is "properly supported") of two ψDO's of orders m, m' is a ψDO of order $m + m'$.
2. The principal symbol of the product of two ψDO's is the product of the principal symbols of the factors.
3. A ψDO of order ≤ 0 is bounded. For order < 0 it is compact.
4. A ψDO of order less than $-n$ on a manifold of dimension n is trace class.
5. If A is a ψDO on a manifold, and if ϕ_j and ψ_j are as above, then A may be written

$$A = \sum_j \psi_j A \phi_j + A'$$

with $A' \in L^{-\infty}$ (smoothing operator).

Remark 2.1. Note that the classical ψDO's form an algebra which is an example of a more abstract object which usually is referred to as Weyl algebra. According to [17], it is a Weyl algebra corresponding to the symplectic cone $Y = T^*M \setminus \{0\}$ ($\{0\}$ the zero section), with its standard symplectic form ω and $\mathbb{R}^+$-action $\rho_t(x, \xi) = (x, t\xi)$. That is, Y is an $\mathbb{R}^+$-principal bundle such that $\rho_t^*\omega = t\omega$. The properties listed above, however, are only part of the conditions assumed in [17, 2., A.1.-E.].

2.1.2. *Definition of the Wodzicki residue.* There are at least two equivalent defini-tions of the Wodzicki residue: As a residue of a certain ζ-function and as an integral of a certain local density [44], [24]. We take as starting point the second definition which can be used most directly for writing classical gauge field Lagrangians. The first definition will show up in the second proof of Connes theorem.

Definition 2.4. Let M be an n-dimensional compact Riemannian manifold. Let T be a classical pseudodifferential operator of order $-n$ acting on sections of a complex vector bundle $E \longrightarrow M$. The Wodzicki residue of T is defined by

$$Res_W(T) = \frac{1}{n(2\pi)^n} \int_{S^*M} \mathrm{tr}_E \sigma_{-n}(T)\mu, \tag{2.11}$$

where $\sigma_{-n}(T)$ is the principal symbol of T, S^*M is the cosphere bundle $\{\xi \in T^*M : \|\xi\|_g = 1\}$ and μ is the volume element defined by a multiple of the canonical contact form on T^*M. tr_E is the natural pointwise trace on $\pi^*(End(E))$.

The form μ is defined as $\mu = \frac{(-1)^{\frac{n(n+1)}{2}}}{(n-1)!} \alpha \wedge (d\alpha)^{\wedge(n-1)}$, where α is the canonical 1-form on T^*M, $\alpha = \sum_i \xi_i dx^i$ in local coordinates.

Also Res_W is defined for classical ψDO of any order, using the same formula with p_{-n} instead of σ_{-n} for integer order $m \geq -n$, and putting $Res_W = 0$ else.

It should be noted that the Wodzicki residue can be defined without using the Riemannian structure [44]: One starts defining for a ψDO T on a chart domain in $\mathbb{R}^n$ a matrix-valued local density

$$res_x(T) = (\int_{\|\xi\|=1} p_{-n}(x,\xi)|\bar{d}\xi|) |dx|,$$

where $\bar{d}\xi = \sum_i (-1)^i \xi_i d\xi_1 \wedge \cdots \wedge \hat{d\xi_i} \wedge \cdots \wedge \xi_n$ is the normalized volume form on the standard Euclidean sphere $\|\xi\| = 1$ and dx is the standard volume form in the chart coordinates. Note that $\bar{d}\xi dx = \mu$. Then one shows that this has good functorial properties, i.e. is indeed a density (an absolute value of an n-form) on M, and defines

$$Res_W(T) = \frac{1}{n(2\pi)^n} \int_M \mathrm{tr}\, res_x(T).$$

Due to the homogeneity property of $p_{-n}(x,\xi)$ (using the Euler formula), $p_{-n}(x,\xi)\bar{d}\xi$ is a closed form, thus $\|\xi\| = 1$ can be replaced by any homologous $n-1$-surface, in particular by any sphere $\|\xi\|_g = 1$ with respect to a chosen Riemannian metric on M. This leads to formula (2.11) used above. Thus, $Res_W(T)$ does not depend on the choice of the Riemannian metric defining the cosphere bundle. It may, however, depend on the metric through a metric-dependence of T.

Remark 2.2. The residue Res_W defined above coincides, up to a universal factor which depends only on $\dim(M)$, with the residue defined in [17, DEFINITION 6.1].

Properties of the Wodzicki residue (see [44, 24] and [17, PROPOSITION 6.1]):
1. Res_W is a linear (in general not positive) functional on classical ψDO's.
2. Res_W is a trace on the algebra of classical ψDO's.
3. It is the only trace if M is connected, $\dim(M) > 1$.
4. Res_W vanishes on operators of order $< -n$ or noninteger.

2.2. Connes' trace theorem.

2.2.1. *Formulation of Connes' trace theorem.* We are now ready to formulate the famous trace theorem [3].

Theorem 2.2. *Let M be a compact Riemannian manifold of dimension $\dim(M) = n$, let $E \longrightarrow M$ be a complex vector bundle over M, and let T be a classical pseudodifferential operator of order $-n$ on $\Gamma(E)$. Then*

(i) *The extension of T to the Hilbert space $\mathcal{H} = L^2(M, E)$ belongs to the ideal $L^{1,\infty}(\mathcal{H})$.*

(ii) *The Dixmier trace $\mathrm{Tr}_\omega(T)$ coincides with the Wodzicki residue,*

$$\mathrm{Tr}_\omega(T) = Res_W(T) = \frac{1}{n(2\pi)^n} \int_{S^*M} \mathrm{tr}_E \sigma_{-n}(T)\mu. \tag{2.12}$$

As a consequence, $\mathrm{Tr}_\omega(T)$ does not depend on the choice of the functional ω in this case.

The following two parts will be devoted to proofs of this theorem exclusively. Two variants of proving will be presented:

In the first variant we are following roughly the line of the original arguments given in [3], but see also [26] and [14] for some details [12], and the special case of scalar operators is dealt with, essentially. Thereby, to keep short, in some parts the proof will be left a bit sketchy. However, in any case it will be at worst detailed enough to convince the reader of the validity of Connes' trace theorem for the example of the scalar operator $(1 + \Delta)^{-n/2}$ on special compact manifold like the n-torus $\mathbb{T}^n$ or the n-sphere $\mathbb{S}^n$, respectively (Δ is the Laplacian there).

The second variant of proving will be based on an application of Corollary 1.7 and formula (1.10b), together with some of the knowledge gained while proving Connes' theorem in one of the above mentioned special cases which were completely treated in course of the first variant of the proof. Thereby, according to the arguments found in [17], we firstly learn that (1.10b) gets applicable, and secondly see that a complete proof only requires to consider this formula explicitely for a non-trivial example (i.e. one with non-vanishing Dixmier-trace). We emphasize that it is due to the pecularity of this second line of argumentation that along with a special case then validity of the theorem in its full generality – not only for scalar operators – can be concluded.

2.2.2. *On the proof of Connes' trace theorem.* The idea is to see first that the theorem is true if it is true on one manifold and then to prove it on a manifold one likes, e. g. $\mathbb{T}^n$ or $\mathbb{S}^n$.

First, the theorem is true on a manifold M globally iff it is true locally. This is due to property 5. of ψDO's given above and the fact that smoothing operators are in the kernels of both Res_W and Tr_ω. Now one can transport the local situation, using a local diffeomorphism, to a local piece of another manifold M'. Both sides of the desired equation do not change under this transport. Using now again the above local-global argument, we can think of this local operator as part of a global operator on M' (gluing by means of a partition of unity). Thus, if the theorem is true on M', it must also be true on M, otherwise we would have a contradiction.

Let us prove point (i) of the theorem for scalar operators on $\mathbb{T}^n$. First we show $T \in L^{1,\infty}$ for any ψDO of order $-n$ on $\mathbb{T}^n$. The Laplacian Δ (with respect to the standard flat metric on $\mathbb{T}^n$) is a differential operator of order 2, therefore $(1+\Delta)^{-n/2}$ is a ψDO of order $-n$, and T can be written in the form $T = S(1+\Delta)^{-n/2}$, where S is a ψDO of order 0, therefore bounded. Since $L^{1,\infty}$ is an ideal, it is sufficient to see $(1+\Delta)^{-n/2} \in L^{1,\infty}$.

For the proof we need yet a little result from the general theory of compact operators. Suppose $x \geq \mathbf{0}$ is compact but not of finite rank. Let $\lambda_1 > \lambda_2 > \ldots > 0$ be the ordered sequence of the non-zero eigenvalues of x, with multiplicity m_k for λ_k. Then, for each integer $t \in [0, m_{k+1}]$, $k > 2$, let us consider

$$\gamma_{\{\sum_{j \leq k} m_j + t\}}(x) = \frac{\sum_{j \leq k} \lambda_j\, m_j + \lambda_{k+1}\, t}{\log\{\sum_{j \leq k} m_j + t\}}\,, \qquad (2.13a)$$

which yields all terms $\gamma_n(x)$ of the sequence (1.3a) with $\sum_{j \leq k} m_j \leq n \leq \sum_{j \leq k+1} m_j$. From (2.13a) with the help of the properties of the logarithm one then easily infers

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[12]We are very indebted to B. Crell, Leipzig, who kindly placed to our disposal his manuscript [5] and the reading of which was strongly facilitating our understanding of some of the peculiarities of Connes' approach towards formula (2.12).

that for the mentioned t's the following estimate holds:

$$c_{k+1}^{-1}\, \gamma_{\{\sum_{j\le k} m_j\}}(x) \le \gamma_{\{\sum_{j\le k} m_j+t\}}(x) \le c_{k+1}\, \gamma_{\{\sum_{j\le k+1} m_j\}}(x)\,, \qquad (2.13\text{b})$$

with $c_{k+1} = 1+\{\log(1 + (m_{k+1}/\sum_{j\le k} m_j))/\log \sum_{j\le k} m_j\}$. In view of the structure of the latter coefficients from (2.13b) then the following and often useful auxiliary criterion can be seen to hold.

Lemma 2.1. *Let $x \in \mathcal{C}\mathsf{B}(\mathcal{H})_+\backslash\mathcal{F}\mathsf{B}(\mathcal{H})$. Suppose $\lim_{k\to\infty} \gamma_{\{\sum_{j\le k} m_j\}}(x)$ exists. If the sequence $\{m_{k+1}/\sum_{j\le k} m_j : k \in \mathbb{N}\}$ is bounded, then also $\lim_{n\to\infty} \gamma_n(x)$ exists.*

Now we are ready to start our considerations around $(\mathbf{1}+\Delta)^{-n/2}$. The spectrum of the Laplacian $\Delta = -\sum_{i=1}^n \partial_i^2$ on $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is pure point, consisting of the values $\sum_i k_i^2$, $k_i \in \mathbb{Z}$. The corresponding eigenfunctions are e^{ikx}, $k \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$. The multiplicity of an eigenvalue λ of Δ is $m(\lambda) = \#\{k \in \mathbb{Z}^n | \sum_i k_i^2 = \lambda\}$. From this follow analogous facts for the operator $(1+\Delta)^{-n/2}$. Let m_k be the multiplicity of the k-th eigenvalue of the latter. Let

$$\tilde{\gamma}_R((1+\Delta)^{-n/2}) = \frac{\sum_{1+\|k\|^2\le R^2}(1+\|k\|^2)^{-n/2}}{\log N_R'}\,,$$

where N_R' is the number of lattice points in $\mathbb{Z}^n$ with $1+\|k\|^2 \le R^2$. By construction it is easily seen that convergence of $\{\tilde{\gamma}_R((1+\Delta)^{-n/2}) : R \in \mathbb{R}_+\backslash\{0\}\}$ as $R \to \infty$ implies the limit $\lim_{k\to\infty} \gamma_{\{\sum_{j\le k} m_j\}}((1+\Delta)^{-n/2})$ of the considered subsequence of the sequence (1.3a) to exist (in which case then both limits have the same value). It is not hard to see that for geometrical reasons with the above multiplicities also the other condition in the hypotheses of Lemma 2.1 is fulfilled; this e.g. can be concluded as a by-result from our estimates given below and relating the asymptotic behavior of the ratio between the surface of an n-sphere to the volume of the n-ball of the same radius R within $\mathbb{R}^n$. Hence, in view of Lemma 2.1 the conclusion is that if the limit

$$\lim_{R\to\infty} \tilde{\gamma}_R((1+\Delta)^{-n/2}) = \lim_{R\to\infty} \frac{\sum_{1+\|k\|^2\le R^2}(1+\|k\|^2)^{-n/2}}{\log N_R'}$$

can be shown to exist, then by Theorem 1.2 (2) it has to equal $\mathrm{Tr}_\omega(1+\Delta)^{-n/2}$ (independent of ω). Also, it is not hard to see that the latter limit exists if

$$\lim_{R\to\infty} \frac{\sum_{\|k\|\le R}\|k\|^{-n}}{\log N_R}$$

exists, where N_R is the number of lattice points with $\|k\| \le R$, in which case then both limits yield the same value. We prove that the latter limit exists, computing its value. It is well known [41] that

$$N_R = V_R + O(R^{\frac{n-1}{2}}),$$

where $V_R = \frac{\Omega_n}{n} R^n$ (volume of the ball of radius R in $\mathbb{R}^n$), $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ (area of the sphere S^{n-1}). Neglecting terms of lower order in R, we have

$$N_R = V_R + \ldots = \frac{\Omega_n}{n} R^n + \ldots.$$

In order to determine $\sum_{\|k\|\le R}\|k\|^{-n}$ for large R, we first count the number of lattice points in a spherical shell between R and $R + dR$,

$$N_{R+dR} - N_R = V_{R+dR} - V_R + \ldots = \Omega_n R^{n-1} dR + \ldots.$$

Integrating this, we obtain asymptotically for large R

$$\sum_{\|k\|\leq R} \|k\|^{-n} = \Omega_n \int_1^R r^{-n} r^{n-1} dr + \ldots = \Omega_n \log R + \ldots .$$

Together with $\log N_R = n \log R + \log \Omega_n - \log n + \ldots$ this leads to

$$\lim_{R\to\infty} \frac{\sum_{\|k\|\leq R} \|k\|^{-n}}{\log N_R} = \frac{\Omega_n}{n},$$

i.e.

$$\mathrm{Tr}_\omega (1+\Delta)^{-n/2} = \frac{\Omega_n}{n}. \tag{2.14}$$

It is much easier to determine the Wodzicki residue of $(1+\Delta)^{-n/2}$: the principal symbol of $(1+\Delta)^{-n/2}$ is $\sigma_{-n}((1+\Delta)^{-n/2})(x,\xi) = \|\xi\|^{-n}$, where $\|.\|$ denotes the standard euclidean metric on $\mathbb{R}^n$. Therefore,

$$Res_W((1+\Delta)^{-n/2}) = \frac{1}{n(2\pi)^n} \int_{S^*\mathbb{T}^n} \bar{d}\xi dx = \frac{1}{n(2\pi)^n} \Omega_n \int_{\mathbb{T}^n} dx =$$

$$= \frac{1}{n(2\pi)^n} \Omega_n (2\pi)^n = \frac{\Omega_n}{n},$$

coinciding with the result for the Dixmier trace. Thus, the theorem is already proved for a special operator on $\mathbb{T}^n$.

To prove point (ii) of the theorem, we start with some general remarks about Tr_ω. It is a positive linear functional on the space L^{-n} of ψDO's of order $-n$ with $L^{-n-1} \subset \ker \mathrm{Tr}_\omega$, because elements of L^{-n-1} are trace class (see property 4. above). Since always $L^{-n}/L^{-n-1} \simeq S^{-n}/S^{-n-1}$ is fulfilled[13] it then follows that Tr_ω may be considered as a linear functional on S^{-n}/S^{-n-1}, the space of principal symbols of ψDO's of order $-n$. By restriction, it is also a linear functional on the space of principal symbols of classical ψDO's of order $-n$. On the other hand, this latter space and the space $C^\infty(S^*M)$ coincide, since every element of $C^\infty(S^*M)$ by homogeneity defines a classical principal symbol, ([7], 23.29.11.). Thus, we end up with a linear functional on $C^\infty(S^*M)$. It follows from symbol calculus that this functional is positive (see [5]). Thus, we have a positive distribution, which is always given by a positive measure on S^*M ([6], 17.6.2).

Since an isometry of M gives rise to a unitary transformation of $L^2(M, v_g)$, and the spectrum of an operator does not change under unitary transformations, the Dixmier trace is invariant under isometries. Therefore, the corresponding measure on S^*M is invariant under isometries.

Considering now the case $M = \mathbb{S}^n$, the standard n-sphere with the metric induced from the euclidean metric on $\mathbb{R}^{n+1}$, the group of isometries is $SO(n+1)$, and S^*M is a homogeneous space under the induced action of $SO(n+1)$. It is easy to see that the volume form of the induced Riemannian metric on $S^*\mathbb{S}^n$ is invariant under the action of $SO(n+1)$. Uniqueness of the invariant measure on a homogeneous space shows that the positive measure corresponding to the Dixmier trace must be proportional to the measure given by this volume form v_g,

$$\mathrm{Tr}_\omega(T) = const. \int_{S^*M} \sigma_{-n}(T) v_g.$$

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[13]cf. second footnote on page 18.

It is an easy exercise to show that v_g coincides in this case with the form μ defined above. Moreover, the constant, which neither depends on the operator nor on the Riemannian manifold, is determined by the example of the torus.

Remark 2.3. Note that it follows from Corollary 1.7 and 2.2 that the Wodzicki residue in some cases coincides with a residue of the zeta function $\zeta_T(z) = \operatorname{tr} T^z$ (see also below).

2.2.3. *An alternative proof of the Connes' trace formula.* In this part another derivation of the trace formula (2.12) will be presented. Thereby, the line will be to make the abstract formula (1.10b) directly accessible for Connes' trace formula in the special case of a classical pseudodifferential operator T of order $-n$. Thereby, by some continuity argument, it suffices if the case of positive-definite operators of that kind can be dealt with.

According to the hypotheses in Corollary 1.7 under which (1.10b) is supposed to hold, for this aim it is sufficient to know that the 'right' asymptotic behavior of the spectral values of T holds, and in which case then the extensions of the ζ_T-function of (1.8) have to be analyzed.

Both exercises can be achieved at once and almost without proof by means of V. Guillemin's methods given in [17]. In demonstrating this way towards formula (2.12) we finally will end up with an alternative proof of Theorem 2.2.

Before doing this, we recall the special settings corresponding to the assumptions of Theorem 2.2. In line with these and in accordance with *Remark* 2.1 the Weyl algebra $\mathcal{W}$ of all classical ψDO's corresponding to the symplectic cone $Y = T^*M \setminus \{0\}$ will be considered. We then have the Hilbert space $\mathcal{H} = L^2(M, E)$, which is the completion of the sections $\Gamma(E)$ under a scalar product descending from a symplectic volume element μ on Y and a fibre metric on E. If these are fixed, each classical ψDO T which is at most of order 0 corresponds to a bounded linear operator, and can be identified with its unique bounded linear extension $x = T$ from sections $\Gamma(E)$ onto the whole $L^2(M, E)$. Accordingly, in such case we use the same notation T for both the ψDO and its unique bounded linear extension on all of $L^2(M, E)$.

Now, let x be a bounded linear operator on $\mathcal{H}$. Then, in the special case that x is positive-definite, there exists in a unique way the inverse of x on $\mathcal{H}$, that is, a densely defined, positive-definite self-adjoint linear operator x^{-1} on $\mathcal{H}$, with $x^{-1} x = \mathbf{1}$ and $x\, x^{-1} \subset \mathbf{1}$. In this case we then define P_x to be the n-th (positive) root of this inverse, $P_x = \sqrt[n]{x^{-1}}$.

Especially, if $T \in L_{cl}^{-n}(= \mathcal{W}^{-n}$ in the terminology of [17]) is supposed to be positive-definite, by compactness and in view of the definition of – and the properties coming along with – the term ellipticity, for $x = T$ the operator x^{-1}, and thus also P_x, is positive-definite, self-adjoint, elliptic and of order one.

Now, from [17, (1.1)] one in particular learns that for the asymptotic growth of the singular values of a positive-definite, self-adjoint elliptic differential operator P of order one the Weyl's formula [43] holds. This equivalently says that the singular values behave like $\mu_k(P) \sim l \sqrt[n]{k}$, with some constant l, see [34, (13.18) and Proposition 13.1]. Hence, in the above-mentioned special case $P = P_x$ this asymptotic law amounts to $\mu_k(T) = \mu_k(x) \sim L \cdot k^{-1}$, with some constant $L \geq 0$, for each positive-definite $T \in L_{cl}^{-n}(= \mathcal{W}^{-n})$. Especially, in line with Lemma 1.6 we then have $T \in L^{1,\infty}(\mathcal{H})$, for each such operator. Since $L^{1,\infty}(\mathcal{H})$ is an ideal from this especially $L_{cl}^{-n} \subset L^{1,\infty}(\mathcal{H})$ is seen, which demonstrates that Theorem 2.2 (i) can be equivalently followed also from the main result of [17].

On the other hand, owing to Weyl's asymptotic law, the applicability of formula (1.10b) now will rely on the extension properties of ζ_x from the half-plane $\Re z > 1$ onto $\mathbb{C}$ exclusively. This matter we are going to discuss now.

We start with some preliminary considerations about various existing definitions relating to ζ-functions which can be associated to some positive operator.[14]

Firstly, in accordance with the above and Corollary 1.7 the ζ_x-function as given in (1.8) for compact positive $x \in L_{cl}^{-n} \subset L^{1,\infty}(\mathcal{H})$ is holomorphic in the half-plane $\Re z > 1$. Thus for given fixed $n \in \mathbb{N}$ upon defining $\zeta^x(w) = \zeta_x(z)$ at $w = n(1-z)$, $\Re z > 1$, one gets another complex function $w \mapsto \zeta^x(w)(= \zeta_n^x(w))$ which is holomorphic in the half-plane $\Re w < 0$. Recall that in view of the arguments given in context of (1.8) the operator family $\Re z > 1 : z \mapsto x^z$ consists of trace-class operators. But then, by functional calculus this operator family may be equivalently re-defined at each $w \in \mathbb{C}$ with $\Re w < 0$ and $z = (1 - w/n)$ as $\Re w < 0$, $w \mapsto x P_x^w$. Hence, if another ζ-function

$$\forall\, w \in \mathbb{C}, \Re w < 0 : \ \zeta(x, P)(w) = \operatorname{tr} x P^w \tag{2.15}$$

is defined for bounded-invertible, positive-definite self-adjoint linear operator P, and bounded $x \geq \mathbf{0}$ such that $x P^w \in L^1(\mathcal{H})$ for all w with $\Re w < 0$, in view of the above in the special case of $P = P_x$ we may summarize as follows.

Lemma 2.2. *Suppose $x \in L_{cl}^{-n}$ to be positive-definite. Then, in the special case of $P = P_x$ the complex function $w \mapsto \zeta(x, P)(w)$ is holomorphic in the half-plane $\Re w < 0$, and there $\zeta(x, P)(w) = \zeta_x(z)$ is fulfilled, at $z = (1 - w/n)$.*

Now, from [17, THEOREM 7.4] one also knows that the nuclear dimension of $\mathcal{W}$ is n (the conclusion of Theorem 2.2 (i) being in accordance with this), where n is the dimension of the basic manifold M. Hence and especially, if $x = T \in L^{-n}$ is a classical ψDO ($\in \mathcal{W}^{-n}$ in the terminology of [17]), then for each positive-definite, self-adjoint *elliptic* operator $P \in \mathcal{W}^1$ (that is, P is of order one, and among other facts, has to be bounded-invertible on $\mathcal{H}$, e.g.) the ζ-function $w \mapsto \zeta(T, P)(w)$ of (2.15) may be considered. In fact, as a consequence of positive-definiteness and ellipticity of P and since T is of order $-n$ (which is the negative of the nuclear dimension of $\mathcal{W}$) all the conditions under which (2.15) is to hold are fulfilled, and thus the restriction to the negative half-plane of the trace of the operators $T(w)$, whose operator family is given as $T(w) = T\, P^w \in \mathcal{W}^{w-n} \subset \mathcal{W}^w$ for $w \in \mathbb{C}$, makes sense. Thereby, the mentioned operator family itself is known to possess a canonical property; it is a so-called *holomorphic* family of operators [15].

The latter especially means that the conditions of the hypothesis of [17, THEOREM 7.1] are fulfilled, and then in line with the conclusion of this result $\zeta(T, P)(w)$ has to be holomorphic in the half-plane $\Re w < 0$ and has a meromorphic extension to the whole complex plane, and at $w = 0$ has, at worst, a simple pole. Moreover, according to [17, THEOREM 7.4] the residue of this meromorphic extension depends only on the symbol $\sigma_{-n}(T)$ of T, and has the form

$$\mathfrak{Res}|_{w=0}\, \zeta(T, P) = g_0 \operatorname{Res}(\sigma_{-n}(T)), \tag{2.16a}$$

with a non-zero constant, $g_0 \neq 0$, which depends only on the Weyl algebra $\mathcal{W}$ under consideration. Now, remind that we are in the special context described

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[14]We have to thank H. Upmeier, Marburg, who outlined to us some of the relevant details.

[15]For the precise definition and basic properties around *ellipticity* and *holomorphy* for ψDO's of a given order and operator families, respectively, we refer to [17, DEFINITION 2.1, PROPOSITION 4.1, and eq. (3.18)]

in *Remark 2.1*. But then, the cosphere bundle $S^*M = \{\xi \in T^*M : \|\xi\| = 1\}$ appears as the (compact) base of the symplectic cone $Y = T^*M\backslash\{0\}$. Denoting by $\pi' : Y \to S^*M$ the projection of Y, one defines for a homogeneous (of degree $-n$) C^∞-section f of the bundle $\pi^*End(E) \to Y$, according to [17, DEFINITION 6.1],

$$\mathrm{Res}(f) = \int_{S^*M} \mathrm{tr}_E f\tilde{\mu} = \int_{S^*M} \mu_f \,. \tag{2.16b}$$

Here $\tilde{\mu} = \alpha \wedge \omega^{\wedge(n-1)}$ is the volume element defined by a salar multiple of the canonical contact form on T^*M (and thus in accordance with *Remark 2.1* one has $\rho_t^*\tilde{\mu} = t^n\,\tilde{\mu}$, μ_f is the uniquely determined $(2n-1)$-form defined through $\mathrm{tr}_E f\tilde{\mu} = \pi'^*(\mu_f)$ and tr_E is the natural pointwise trace on $\pi^*(End(E))$. As mentioned above there is a basic fact saying that homogeneous C^∞-sections of $\pi^*End(E) \to Y$ in our case exactly yield all the principal symbols to classical ψDO's of order $-n$.

Thus, the above-mentioned conclusions about the possibility of a meromorphic extension of $\zeta(T, P)$ and its singularity structure at $w = 0$ apply with $P = P_T$. In view of Lemma 2.2 and as a consequence of the just said, upon changing the complex variable w into z in accordance with $w = n(1 - z)$ we will see that analogous facts hold in respect of ζ_T and at $z = 1$, accordingly. That is, ζ_T possesses a meromorphic extension into the whole z-plane, with a simple pole at $z = 1$, at worst. Having in mind this, and taking into account that from $w = n(1 - z)$ a geometric factor $1/n$ arises while passing from the residue of the one extension at $w = 0$ to the residue of the transformed extension at $z = 1$, in view of (2.16a)–(2.16b) we then may summarize as follows:

Corollary 2.1. *Let $T \in L_{cl}^{-n}$ be positive-definite. The holomorphic function $\zeta_T(z)$ has a meromorphic extension from the half-plane $\Re z > 1$ into the whole complex plane with, at worst, a simple pole at $z = 1$. The residue of the extension obeys*

$$\mathfrak{Res}|_{z=1}\, \zeta_T(z) = (g_0/n) \int_{S^*M} \mathrm{tr}_E \sigma_{-n}(T)\mu \,,$$

with a constant $g_0 \neq 0$ which does not depend on the special operator T.

Foremost, according to Corollary 1.7 and by the above-mentioned asymptotic spectral properties the Corollary guarantees that formula (1.10b) can be applied for positive-definite classical ψDO's of order $-n$, with the result that

$$\mathrm{Tr}_\omega(T) = (g_0/n) \int_{S^*M} \mathrm{tr}_E \sigma_{-n}(T)\mu \tag{2.16c}$$

has to be fulfilled, for each positive-definite T of order $-n$. Also, in order to fix the constant g_0 it obviously suffices to deal with one particular case of such an operator. Moreover, once more again according to the local-global and the M-to-M' arguments, which we have already mentioned at the beginning of 2.2.2 while proceeding the first variant of the proof of Theorem 2.2, we have to conclude that the constant g_0 within (2.16c) has to be the same, in each case of an n-dimensional compact manifold M. Hence, we may content with the known result for $T = (\mathbf{1} + \Delta)^{-n/2}$ on the n-torus $M = \mathbb{T}^n$. According to our calculations therefore $g_0 = 1/(2\pi)^n$ has to hold, and then (2.16c) will yield that (2.12) has to be valid, for each positive-definite T of order $-n$ on an arbitrary compact n-dimensional manifold M. ¿From this the validity for all positive T of order $-n$ can be concluded, since for fixed positive-definite T_0 of order $-n$ and each positive T

the family $T(\varepsilon) = T + \varepsilon T_0$, $\varepsilon > 0$, consists of positive-definite ψDO's of order $-n$, for which according to the above the assertion of Connes formula holds. In fact, according to Lemma 1.5 one knows $\mathrm{Tr}_\omega(T(\varepsilon)) = \mathrm{Tr}_\omega(T) + \varepsilon\,\mathrm{Tr}_\omega(T_0)$. On the other hand, the map $A \mapsto \sigma_{-n}(A)$ between ψDO's of order $-n$ and their principal symbols is a homomorphism, and therefore also $\sigma_{-n}(T(\varepsilon)) = \sigma_{-n}(T) + \varepsilon\,\sigma_{-n}(T_0)$. Hence, since the expression on the right-hand side of (2.16c) obviously is a linear form with respect to the $\sigma_{-n}(T)$-variable, the validity of (2.12) in the general case can be obtained simply via the just mentioned linearity and upon taking the difference between a relation of type (2.16c), taken at one particular $T(\varepsilon)$, for some $\varepsilon > 0$, and a multiple of the relation of type (2.16c) at T_0 with ε.

2.3. **Classical Yang-Mills actions.** Here we make some remarks about the construction of the bosonic part of classical (pure) gauge field actions in terms of the Dixmier trace and the classical Dirac operator. This was considered in more detail in the lectures by R. Holtkamp and K. Elsner/H. Neumann. We will make use of the fact that the de Rham algebra of exterior forms is isomorphic to the differential algebra $\Omega_D(C^\infty(M))$ coming from the classical spectral triple $(\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(M,S), D)$, D the Dirac operator on the compact n-dimensional Riemannian spin manifold M, S the spinor bundle (see the lecture by M. Frank). The representation π of $\mathcal{A}$ on $\mathcal{H}$ is given by sending $f \in \mathcal{A}$ to the operator of multiplication with the function f.

2.3.1. *The classical Dirac operator and integration on manifolds.* First, we notice

Proposition 2.1. *Consider* $f \in C^\infty(M)$ *as left multiplication operator on* $L^2(M,S)$. *Then*

$$\mathrm{Tr}_\omega\left(f|D|^{-n}\right) = \frac{1}{c(n)} \int_M f\ v_g, \tag{2.17}$$

where v_g *denotes the Riemannian volume element,* $c(n) = 2^{n-[n/2]-1}\pi^{n/2}n\Gamma(n/2)$, *and* Tr_ω *is the Dixmier trace with respect to any invariant mean* ω.

Proof. (see [26], p. 98) The principal symbol of the Dirac operator is $\gamma(\xi)$ (Clifford multiplication on spinors), thus D is a first order (elliptic) ψDO. Multiplication with f is a zero order operator, therefore $f|D|^{-n}$ is a ψDO of order $-n$. Its principal symbol is $\sigma_{-n}(x,\xi) = f(x)\|\xi\|^{-n}1_{2^{[n/2]}}$, where $1_{2^{[n/2]}}$ is the identity map of the fibre S_x of S. This principal symbol reduces on the cosphere bundle S^*M to $f(x)1_{2^{[n/2]}}$. Thus, Theorem 2.2 gives

$$\mathrm{Tr}_\omega\left(f|D|^{-n}\right) = \frac{1}{n(2\pi)^n} \int_{S^*M} \mathrm{tr}_S(f(x)1_{2^{[n/2]}})dx\bar{d}\xi$$

$$= \frac{2^{[n/2]}}{n(2\pi)^n} \int_{S^{n-1}} \bar{d}\xi \int_M f(x)dx.$$

The area $\int_{S^{n-1}} \bar{d}\xi = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ of the unit sphere S^{n-1} leads to the right factor $c(n)$. $\square$

Since $|D|^{-n}$ is in $L^{1,\infty}$ we can define the following inner product on $\pi(\Omega^k\mathcal{A})$:

$$\langle T_1, T_2\rangle_k := \mathrm{Tr}_\omega\left(T_1^*T_2|D|^{-n}\right). \tag{2.18}$$

In order to really have an inner product, one needs some assumptions about $\mathcal{A}$ which are fulfilled in the classical case, but which also hold in more general situations of

spectral triples, see [2, 26, 39].[16] The orthogonal complement with respect to this inner product of the subspace $\pi(d(J_0 \cap \Omega^{k-1})) \subset \pi(\Omega^k \mathcal{A})$ is isomorphic to $\Omega_D^k \mathcal{A}$ (both are images of surjections with the same kernel). Thus, the inner product (2.18) can be tranported to $\Omega_D^k \mathcal{A}$.

Proposition 2.2. *Under the isomorphism between $\Omega_D^k \mathcal{A}$ and $\Gamma(\Lambda_{\mathbb{C}} T^* M)$ the inner product on $\Omega_D^k \mathcal{A}$ is proportional to the usual Riemannian inner product,*

$$\langle \omega_1, \omega_2 \rangle_k = (-1)^k \lambda(n) \int_M \omega_1 \wedge {}^* \omega_2 \tag{2.19}$$

for $\omega_i \in \Omega_D^k \mathcal{A} \simeq \Gamma(\Lambda_{\mathbb{C}} T^* M)$, where

$$\lambda(n) = \frac{2^{[n/2]+1-n} \pi^{-n/2}}{n \Gamma(n/2)}.$$

Proof. We refer to [26], p. 120.   □

2.3.2. *Classical gauge field actions in terms of Dixmier-trace.* Now, in usual gauge theory, the gauge field F may be interpreted as a two-form with values in the endomorphisms of a vector bundle E over M (curvature of a connection). (Such vector bundles typically arise as bundles associated to a principal bundle with the group of inner symmetries as structure group). The (pure) gauge field action is then constructed by combining the scalar product on the right-hand side of (2.19) with a (fiberwise) product and trace of the endomorphisms,

$$YM(\nabla) = const. \int_M tr(F \wedge *F).$$

By (2.19) this can be written equivalently in terms of the differential algebra $\Omega_D^k \mathcal{A}$ and the scalar product there.

Moreover, it is almost obvious from the definition of the inner product that the classical YM action can be obtained as an infimum over a "universal" YM action defined over universal connections (which are elements of $\Gamma(EndE) \otimes_{C^\infty(M)} \Omega^2(\mathbb{C}^\infty(M))$. More precisely [4], [26], one shows that

$$1 \otimes \pi : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1 \mathcal{A}$$

($\mathcal{E} := \Gamma(E)$) gives rise to a surjection from universal connections to usual connections. If θ is the curvature of a universal connection ∇_{un}, one defines

$$I(\nabla_{un}) = \mathrm{Tr}_\omega \left(\{1 \otimes \pi\}(\theta)^2 |1 \otimes D|^{-n} \right).$$

and finds

$$YM(\nabla) = const. \inf\{I(\nabla_{un}) | \pi(\nabla_{un}) = \nabla\}.$$

Thus the classical Yang-Mills action can be entirely written in terms of objects which have a straightforward generalization to the noncommutative situation.

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[16]Thanks to J. Várilly for pointing out this fact to us.

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